Spinors in classical mechanics

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Spinori u Klasičnoj Mehanici

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Abstract

Spinors were first mentioned by Elie Cartan (1913) and were introduced in physics by Dirac. Though it is conventionally assumed spinor wave functions are a consequence of quantum weirdness or relativity, and while on closer inspection they can be seen as having topological origin, it can be shown that the fundamental reason for the appearance of spinorial objects in quantum mechanics is purely geometrical. Spinors are naturally described in the language of Clifford algebra which Clifford himself originally intended to be a universal geometric language. To that end, the Kepler problem of classical mechanics is solved in a Newtonian geometric spirit albeit in a modern algebraic formulation, setting the stage for the formulation of the spinor equation of motion which linearizes and regularizes (removes singularity) the inverse square force equation of motion with universal solutions yielding new applications to perturbation theory. This also leads to new insights and interpretations of quantum mechanical observables, and the geometric nature of fermionic spin 1/2 wave functions.

1 Introduction

"For geometry, you know, is the gate of science, and the gate is so low and small that we can only enter it as a little child. " -William Kingdon Clifford

$a \cdot (b \wedge c) = a \cdot bc - a \cdot cb$

The zeroth law of Newtonian physics is that the dynamics takes place in a (3-d)Euclidean space. To better understand the motions and interactions of physical objects, it is important to understand the underlying structure upon which all motion is set. Physical space can be modeled by the Euclidean vector space where trajectories of particles are described by a position vector $\vec{r}$ which can then in practice be parametrized using a particular set of direction vectors (for example the direction of initial velocity and gravitational field in case of projectile motion). Another, perhaps more natural way is to describe motion as a scaling(dilation) and rotation of the initial position vector. As we shall see, this is where spinors come in, but with an interesting "twist", as spinor rotation operators need to be parametrized by half angles, revealing a big hint to their rotational transformation properties. Ordinarily in 3 dimensions describing kinematics and dynamics by rotation operators is not a trivial matter, as basic Euclidean vectors don't seem to come equipped to handle such a formulation. At most, one can describe more elementary operations, namely reflections and corresponding reflection operators $R(a, n)$ which subtract components of vectors parallel to normals of reflecting surfaces thereby producing mirror images of initial vectors.
But such descriptions of rotation algebra are computationally cumbersome particularly when it comes to compositions, which consequently is how one can get a rotation naturally, by composing 2 reflections resulting in a rotation by twice that of the angle between the normals of reflecting surfaces. The fact that reflections can be seen as "atoms" of rotations, as will be shown later, is very important for the spin half transformation laws, but in standard linear algebra, it is not a very clear result though by dropping down to 2 dimensions a simpler way to do rotations can be found.
Mathematicians took a long time to make peace with complex numbers which were a nuisance that kept coming up out of nowhere in various geometric and algebraic problems. However, those "imaginary" numbers offer a simple representation of the algebra of a real 2D Euclidean space. The unit imaginary i can be thought of as a 90° rotation operator in the complex xy plane, which composed with itself gives $ii = i^2 = -1$, which is the 180°operator. Another multiplication with i gives a 270°rotation and so on. In this way, complex numbers naturally represent the algebra of plane geometry.

Algebra of geometry

This line of thought though was not easy to extend to higher dimensions, but an important step towards this goal has been made by Hermann Grassmann and his algebra of extension in which vectors could both be interpreted geometrically as directed line segments and projectively as relations between points[6]. This was done using the anti-commutative outer product which complemented the standard symmetric inner product. The outer product can be interpreted geometrically as a directed area because two non-collinear vectors span an area. Dually it is the algebraic expression of the geometric relations between the parallelogram area and its base and height (figure 1).
If the product of two vectors is to represent the spanned area, it has to be zero for all collinear vectors which implies antisymmetry of the outer product for all vectors a, b,c $\in \mathbf{E(n)}$:

$a \wedge d = a \wedge (\alpha a) = \alpha a \wedge a = 0$ iff $a \wedge a = 0 = (b + c) \wedge (b + c) = b \wedge b + c \wedge c + b \wedge c + c \wedge b = 0$

To complete his algebra Grassmann envisioned multiplication of the form a/b b =a which "rotates" a into b by the angle $\theta(a, b)$, provided a and b are of the same magnitude. Then a/b could be considered a rotation operator and if $\theta(a, b) = 90°$, $(a/b)^2 b = -b$ which means $(a/b)^2 = -1 = i^2$. This then yields the natural exponential parametrization for the rotation operator $a/b = e^{i\theta} = i \sin \theta + cos\theta$ which acts on vectors in a plane. However, he was not able to generalize this method to apply to rotations in space. At around the same time Hamilton discovered quaternions, a 4-dimensional algebra of complex numbers that fully described rotations in 3-dimensional space but it was not clear at the time how it would fit in the algebra of Grassman, Gibbs, or other mathematical systems of the time.

Geometry of algebra

It was the great English mathematician William Clifford who managed to understand Grassmann's great insights and expanded on his work by giving an expression for a product of two quaternions by vector products, the scalar product, and the outer product[13]. The (inner) scalar product of vectors gave the "spatial algebra" its metrical properties needed for distance and angle relations between lines, planes, and volumes, while the outer product ensured that 2 and higher dimensional objects(planes, volumes) had vector-like properties of direction and orientation needed for operational algebraic manipulations.
Clifford saw that the noncommutative product ab of two vectors can unify inner and outer products, and used it as foundation for the development of a universal algebra for geometry ; $ab = \frac{1}{2}(ab+ba)+\frac{1}{2}(ab-ba)$

$$a \cdot b \equiv \frac{1}{2}(ab + ba) = b \cdot a = \frac{1}{2}(ba + ab)$$

$$a \wedge b \equiv \frac{1}{2}(ab - ba) = -b \wedge a = \frac{1}{2}(ba - ab)$$

From distributivity and the fact that the square of any vector is real, it follows that the symmetric (commutative) product of two vectors is also a scalar(real).

$$a \cdot b \equiv \frac{1}{2}(ab + ba) = \frac{1}{2}((a + b)^2 - a^2 - b^2))$$

This is equivalent to the standard inner product.
Thus such non-commutative numbers are indeed Euclidean vectors as they form an inner product space $\vec{\mathbf{E}}(n)$ over the reals, which in turn defines the geometry of Euclidean space through free and transitive actions of the (free) vectors on the abstract affine set of points ;
as per the standard Kleininian(symmetry group) definition of Euclidean space $E(n) \equiv (\{p\}, \vec{\mathbf{E}}(n))$, with transitive actions on points by vectors defined as ; $b + \vec{v} = c, \vec{v} = c - p, d(c, b) = \sqrt{\vec{v} \cdot \vec{v}}$, where b,c $\in \{p\}$ with vectors acting as geometric transformations i.e. group actions (translations) on the set of points $\{p\}$.

The inner product has a direct geometrical interpretation as orthogonality is implied for vectors whose inner product is zero. It can in turn also be seen as the algebraic realization of the (geometric) perpendicular projection of line segments as it designates algebraically that the two lines are orthogonal if the inner product of the vectors representing them is zero.
In this the geometric concept of vectors as directed magnitudes is naturally realized algebraically, as their geometric product is a measure of their relative direction. This on the other hand is just a natural completion of the(multi-linear) algebra of vectors by the general noncommutative product that has a direct geometric interpretation; if vectors commute they are collinear, if they anticommute they are orthogonal.

More generally,
An oriented n-volume form is then embodied in the antisymmetric part of a non-symmetric (Clifford) product of n vectors :

$$V(n) = \langle a_1 a_2 ... a_n \rangle = a_1 \wedge a_2 \wedge ... a_n$$

As outer products of pairs of vectors give oriented areas, an outer product of three vectors results in an oriented volume, and so on.
For an n-dimensional vector space, these outer products on n vectors are exactly the pseudoscalars contained in its geometric algebra, which is the algebra of its subspaces encoded in the structure generated by the iterative application of the geometric (noncommutative, associative, and distributive) product.
In this, we have an intrinsic way to utilize the canonical structure of subspaces for computation, a key thing missing in the theory of (symmetric) inner product spaces.
In the following chapter Clifford's algebra will be developed systematically in a modern language to develop a purely geometric calculus with applications in physics which also naturally contains complex numbers and quaternions as real spinors in 2 and 3 dimensions respectively.

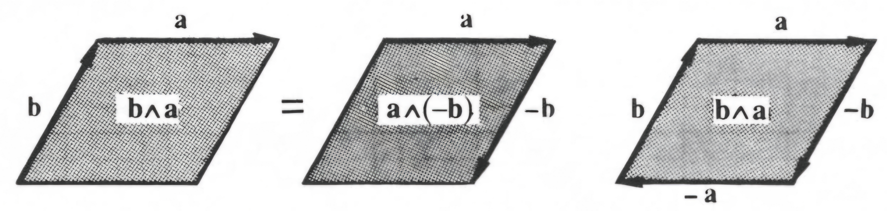

Figure 1: *Figure 1 illustrates the properties of the outer product of vectors [1]*

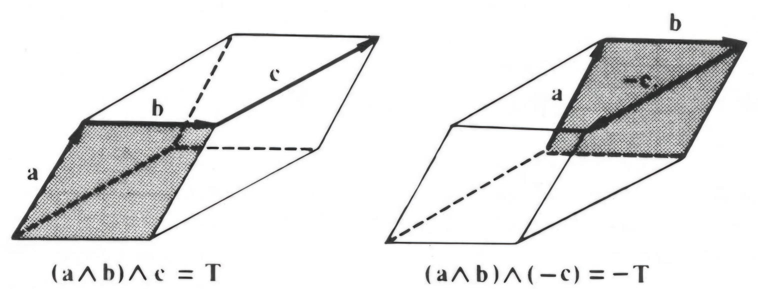

Figure 2: *Outer products of 3 vectors produce oriented parallelepipeds [1]*

2 Clifford algebra

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."
- Michael Francis Atiyah

Let V be an n-dimensional real linear space. Clifford algebra Cl(V) is the graded associative division algebra generated by the left and right distributive, associative Clifford product Cl(V)xCl(V)→Cl(V); $ab \to c$ with the two additional "grading properties" ;

For any $\lambda \in R$; λg=gλ, for any g ∈ Cl(V),

For any vector $v \in V : vv = v^2 = |v|^2$ where $|v|$ is a unique positive real number corresponding to the absolute value of a vector.

With scalars (reals) being labeled as grade 0 elements and vectors grade 1 (1-blades) from the axioms above higher-order grades can be found and labeled as follows :

$$aa_n = \frac{1}{2}(aa_n - (-1)^n a_n a) + \frac{1}{2}(aa_n + (-1)^n a_n a) \qquad (0)$$

$$a \wedge a_n \equiv \frac{1}{2}(aa_n + (-1)^n a_n a) \qquad (1)$$

$$a \cdot a_n \equiv \frac{1}{2}(aa_n - (-1)^n a_n a) \qquad (2)$$

with (1) giving a n+1 grade element or (n+1) blade.

It follows then that the interior product (2) is a grade-lowering operation. It is more general than a metric and it can have non-metrical, signature-independent interpretations, but unlike exterior and geometric products, it is not associative.

To reduce the number of braces, as a convention the interior and exterior products are given priority over the fundamental (Clifford) product.

The above also implies that n-blades are antisymmetric parts of products of more than two vectors, with a direct geometric interpretation of oriented n-volume elements. Orthogonality and collinearity relations between higher-grade blades are also determined by inner and outer products as was the case for vectors (1-blades).
General elements (of mixed grade) are called (geometric) numbers or multivectors. Multivectors that are linear combinations of n-blades are general elements of the n^{th} grade called n-vectors. A vector space is n-dimensional if n-blades are the nonvanishing elements of the highest grade. Such blades are called pseudoscalars as they are equal up to a real scale factor and are thus 1-dimensional numbers as are reals (scalars).

From (0),(1),(2) it follows that :

$ab = a \wedge b + a \cdot b$

Which relates the interior and exterior products of any two 1-vectors a and b additively, and from (2) their multiplicative relation can be derived as follows ;

$abc = -bac + 2a \cdot bc$

$= -b(-ca + 2c \cdot a) + 2a \cdot bc$

$\Rightarrow a \cdot bc - a \cdot cb = \frac{1}{2}(abc - bca)$

$= \frac{1}{2}(ab \wedge c - b \wedge ca) = a \cdot (b \wedge c)$ from (1),(2)

$\Rightarrow a \cdot (b \wedge c) = a \cdot bc - a \cdot cb$ (3)

Thus the interior product of a 2-blade and a vector produces a vector.

The complement of the projection operator P(a, b)=$a \cdot bb^{-1} = a \cdot \hat{b}|b|\hat{b}|b|^{-1} = a \cdot \hat{b}\hat{b}$ -(3a) ;

Is the perpendicular rejection operator R(a,b) that outputs the orthogonal component of a vector to a particular direction. This follows from (3) :

R(a,b)=$a \wedge bb^{-1} = a - a \cdot bb^{-1} = a - a \cdot \hat{b}\hat{b}$ (3b)
These will be important for handling reflections and rotations of vectors.

General projection and rejection operators giving parallel and orthogonal components to higher grade elements representing n-dimensional spaces(planes, volumes, etc) follow from (0),(1),(2) :

$a = aa_n{a_n}^{-1} = a \cdot a_n{a_n}^{-1} + a \wedge a_n{a_n}^{-1} = P(a, a_n) + R(a, a_n)$

$P(a, a_n) = a \cdot a_n{a_n}^{-1} = a_\parallel$ (3c)

$R(a, a_n) = a \wedge a_n{a_n}^{-1} = a_\perp$ (3d)

With the inner and outer products combined multiplicatively in (3) another, an operational, interpretation of the outer products of 2 vectors becomes available; it is the simplest canonical generator of rotations in a plane ;

Taking another contraction :
$a \cdot (a \cdot (b \wedge c)) = a \cdot ((a \cdot b)c - (a \cdot c)b) = (a \cdot b)(a \cdot c) - (a \cdot c)(a \cdot b) = 0$ (4)

This proves that the inner product of a 2-blade with a vector produces a vector orthogonal to the original vector, in the plane represented by the 2-blade (spanned by vectors whose outer products are blades collinear with the 2-blade) which is also rotated by 90°. If the outer product of a vector and a 2-blade is zero they are collinear and their inner product is equal to the associative geometric product.

Thus we get a rotational operator in a plane acting and composing simply and efficiently through the fundamental product of the algebra.
In this way, unit blades rotate vectors in their planes by $\pi/2$ radians, and we will see this in ch.4. in more detail and in the general higher dimensional version.
The composition of two successive rotations in a plane by $\pi/2$ is equal to multiplication by -1. This can be proven by again taking the inner product of the plane rotation operator (unit 2-blade) and the rotated vector, but we can also see this as the direct composition of ($\pi/2$) plane rotation operators, the 2-blades, as they compose through the basic (general or elementary) multiplication, in this case squaring to -1, thus in turn also revealing their "imaginary" nature.

$a^2b^2 = abba = (ab)(ba) = (a \cdot b + a \wedge b)(b \cdot a + b \wedge a) = (a \cdot b)^2 + (a \wedge b - a \wedge b)(a \cdot b) - (a \wedge b)^2 = (a \cdot b)^2 - (a \wedge b)^2$

$\Rightarrow (a \wedge b)^2 = (a \cdot b)^2 - a^2b^2 \qquad (5)$

Also :

$a \cdot (b \cdot (a \wedge b)) = (a \cdot b)^2 - a^2b^2 = (a \wedge b)^2 = (a \wedge b) \cdot (a \wedge b)$ from (3) and (1)

$\Rightarrow (a \wedge b)^2 = a \cdot (b \cdot (a \wedge b)) \qquad (6)$

A more general contraction rule for n-blades with m-blades can also be proven from (1),(2).

To get a normalized (unit) 2-blade the Cauchy-Schwarz rule is needed and it is proven as follows:

$0 < (|a|b - a \cdot b\hat{a})^2 = a^2b^2 - |a|b(a \cdot b)\hat{a} - \hat{a}|a|b(a \cdot b) + (a \cdot b)^2$

$= a^2b^2 - 2|a|\hat{a} \cdot b(a \cdot b) + (a \cdot b)^2 = a^2b^2 - (a \cdot b)^2$

Thus we have : $|a \wedge b| = (a \wedge b)(b \wedge a)$ and $a \hat{\wedge} b = \dfrac{a \wedge b}{|a \wedge b|} = R_{\pi/2} \qquad (7)$

In general, to any element(multivector) m of Clifford algebra Cl(V) there corresponds the absolute value $|m| = ([mm^\dagger]_0)^{1/2}$ where the dagger stands for the The *reversion* operator is defined as follows:

$$(AB)^\dagger = B^\dagger A^\dagger,$$

$$(A + B)^\dagger = A^\dagger + B^\dagger,$$

$$\langle A^\dagger \rangle_0 = \langle A \rangle_0,$$

$$a^\dagger = a \quad \text{if} \quad a = \langle a \rangle_1.$$

It follows that the *reverse* of a product of vectors is

$$(a_1 a_2 \dots a_r)^\dagger = a_r \dots a_2 a_1.$$

The reverse of a bivector $B = a \wedge b$ is given by

$$B^\dagger = (a \wedge b)^\dagger = b \wedge a = -a \wedge b = -B,$$

Using the antisymmetry of the outer product to permute the factors, which can be similarly done for trivectors.

$$a \wedge b \wedge c = -b \wedge a \wedge c = b \wedge a \wedge c = -a \wedge b \wedge c.$$

Thus,

$$(a \wedge b \wedge c)^\dagger = c \wedge b \wedge a.$$

Since R_θ is a rotation in a plane, that is a rotation operator on a 2-d Euclidean vector space, rotation angles will add simply to compositions.

$$\Rightarrow R_\pi = R_{\pi/2+\pi/2} = R_{\pi/2} \circ R_{\pi/2} = R^2{}_{\pi/2}$$

$$= \frac{a \wedge b}{|a \wedge b|}^2 = \frac{(a \wedge b)(a \wedge b)}{(b \wedge a)(a \wedge b)} = \frac{a^2b^2 - (a \cdot b)^2}{(a \cdot b)^2 - a^2b^2} = -1 \qquad (8)$$

$$\Rightarrow \sqrt{-1} = \pm i = \pm \frac{a \wedge b}{|a \wedge b|}$$

Thus ant unit 2-blade is indeed a real imaginary unit i and also a linear operator $i(a) = ia$, a rotation in the plane it's representing, which is the 2-d Euclidean space E(2) =({p},$\vec{\mathbf{E}}(2)$) with $a \wedge b \wedge c = 0$ $a, b \in \vec{\mathbf{E}}$ and $\pm i = a \hat{\wedge} b \in \mathbf{A}(\vec{\mathbf{E}}(2))$ are unit pseudoscalars in the geometric algebra each determining rotation directions opposite to one another.

Rotations are the fundamental symmetries of Euclidean spaces as they preserve the shapes and sizes of all embedded objects including their orientations (unlike reflections which reverse them). To see this it will suffice to prove that reflections operate in a covariant manner in the sense that they preserve the fundamental product of the algebra of the 2-d Euclidean (vector)space.

$$i(x)i(y) = ixiy = ix \cdot iy = -ii \cdot xy = -iixy = xy = x \wedge y + x \cdot y$$

Thus (90°)rotations preserve the geometric product and with that both the outer and inner products, unlike reflections which only preserve orthogonality and inner products (ch. 4.)

As the normalized outer product of two vectors $\frac{a \wedge b}{|a \wedge b|}$ is a 90° plane rotation operator (generator) i, we can expect that the normalized geometric product $\frac{a\ b}{|a\ b|} = \frac{a\ b}{\sqrt{a^2b^2}} = \hat{a}\hat{b}$ is a general plane rotation operator with the outer product $\hat{a} \wedge \hat{b}$ being only its a special case when the vectors in the product are exactly orthogonal.
The geometric product of two vectors, however, is neither a scalar, a vector nor a 2-blade but the object with the most general rotational (transformational) properties called spinor.

A unit spinor $\hat{S} = \frac{ab}{|ab|} = \frac{ab}{\sqrt{a^2b^2}} = \hat{a}\hat{b}$ acts on vectors by geometric multiplication and reduces to a unit 2-blade $i = \frac{a \wedge b}{|a \wedge b|} = \hat{a} \wedge \hat{b}$ (a 90° rotation) when $\hat{a}$ and $\hat{b}$ are orthognal.
It can be confirmed simply that unit spinor multiplication preserves distances between points in E(2), which is represented by vector magnitudes ;

$$|\hat{S}a| = |\hat{a}\hat{b}a| = \sqrt{\hat{a}\hat{b}aa\hat{b}\hat{a}} = |a|$$

Every vector in $\vec{\mathbf{E}}(2)$ a is transformed this way, with the only vector being unmoved (transformed to itself) is the zero vector. Thus unit spinor multiplication in a 2-plane has a single fixed point(vector), or equivalently the eigenvector of this canonical rotation operator is the zero vector, proving that it is a rotation.
To express the rotation operator more explicitly geometric product can be put into the exponential form :

$$ab = a \wedge b + a \cdot b = |a||b|\sin(\theta)\frac{a \wedge b}{|a \wedge b|} + |a||b|\cos(\theta) = |a||b|(\sin(\theta)i + \cos(\theta)) = |a||b|e^{i\theta} \qquad \text{-(9)}$$

Which determines the angle relation between vectors as $\theta(a,b) = \arccos(\frac{a \cdot b}{|a||b|})$ giving more explicit expressions also for the inner (scalar) and outer (bi-vector) parts of the product of two vectors in terms of the radian angle measure, by equating the scalar and bi-vector parts of (9) ;

$$a \cdot b = |a||b|\cos(\theta)$$

$$a \wedge b = |a||b|\sin(\theta)\frac{a \wedge b}{|a \wedge b|}$$

We can confirm this by substitution into (5):

$abba = (|a||b|)^2 = (a \cdot b)^2 - (a \wedge b)^2 = (|a||b|)^2(\sin^2(\theta) + \cos^2(\theta) = (|a||b|)^2$

Thus a rotation through an angle θ $R_\theta(a)$ can be put in the (spinor) operator form :
$R_\theta(a) = \hat{a}\hat{b}a = (\sin(\theta)i + \cos(\theta))a = e^{i\theta}a$, for all vectors a=P(a,i) or equivalently for all vectors in $\in \vec{\mathbf{E}}(2)$

$e^{i\theta}a$ is indeed a rotation through θ by the definition of the angle relation of two vectors:

$\arccos(a \cdot e^{i\theta}a = \cos(\theta)a + i\sin(\theta)a = \arccos(\cos(\theta)) = \theta, \qquad \text{as } a \cdot ia = 0$

Also, $re^{i\theta}$ describes a circle of radius $|r|$ in the complex plane - the circle group S(1) which is just a set of complex numbers of magnitude $|r|$ which in turn also form a group U(1) under complex(geometric multiplication). The circle (lie) group S(1) is diffeomorphic to Spin(2) the group of unit spinors in 2-d represented by oriented unit circle arcs composing additively :
$e^{i\theta}e^{i\alpha}e^{i\beta}\hat{r} = e^{i(\theta+\alpha+\beta)}\hat{r}$

The circle group S(1)≅U(1) can be smoothly parametrised by a real number t : $t \to r(t) = r(t_0)e^{i\theta(t)}$, determining a circular trajectory with (a tangent) velocity $v = \dot{r}(t) = r(t_0)i\dot{\theta}(t)e^{i\theta(t)}$ -where i produces a tangent to the trajectory by rotating r throught 90°,
and acceleration a=$\ddot{r} = -r(t_0)\dot{\theta}(t)^2 e^{i\theta(t)} + r(t_0)i\ddot{\theta}(t)e^{i\theta(t)}$
Which if $\theta(t)$ is linear simply reduces to only the centripetal acceleration :
a=$\ddot{r} = -w^2 r(t_0)e^{iwt}$

This can also be applied the other way around starting from a dynamical equation $\ddot{r}(t) = -|f(r)|\hat{r}(t)$ and deriving the kinematical trajectory which is exactly circular for the following initial conditions :
$|\dot{r}(0)| = |\dot{r}_0| = |v_0| = \sqrt{|r_0 f(r)|} \Leftrightarrow w^2|r| = |f(r)|, v_0 \cdot r_0 = 0$;

$\Rightarrow r(t) = r_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t}$

$\dot{r}(t) = v(t) = \sqrt{|f(r)/r_0|}r_0\hat{r}_0 \wedge \hat{v}_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t} = \sqrt{|f(r)/r_0|}\hat{v}_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t}$

$\ddot{r}(t) = \dot{v}(t) = a(t) = \hat{r}_0 \wedge \hat{v}_0{}^2 \sqrt{|f(r)/r_0|}^2 r_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t} = -|r_0|/|r_0||f(r)|\hat{r}_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t}$

$= -|f(r)|\hat{r}(t)$

With that, one of the most important equations in all of mathematics and physics;

$$e^{i\pi} + 1 = 0$$

gains a fully geometric interpretation.

Note that the real algebra of 3-d Euclidean vector space contains another set of "imaginary" numbers of a higher grade; $\pm\sqrt{-1} = \dfrac{a \wedge b \wedge c}{|a \wedge b \wedge c|} = I$ which can be proven analogously to (5). I is the 3-d unit 'pseudoscalar' (up to a sign) because tri-vectors are 1-dimensional as are scalars (0-vectors), in the case of 3-d generating vector space. It is particularly important for projective(non-metrical) incidence relations, with duality expressed as simply as $a^* = aI^{-1}$.

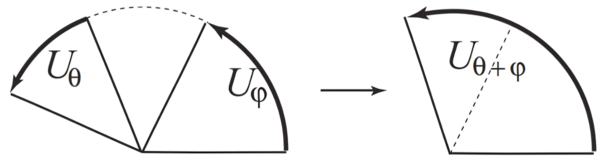

Figure 3: *The composition of 2D rotations is described algebraically by the product of spinors (here designated by the letter U) and represented geometrically by addition of directed circle arcs [12]*

3 Theory in practice

The ponderous instrument of synthesis, so effective in Newton's hands, has never since been grasped by anyone who could use it for such purpose; and we gaze at it with admiring curiosity, as some gigantic implement of war, which stands idle among the memorials of ancient days, and makes us wonder what manner of man he was who could wield as a weapon what we can hardly lift as a burden"

William Whewell on Newton's geometric proofs, 1847

"The geometric calculus differs from the Cartesian geometry in that whereas the latter operates analytically with coordinates, the former operates directly on the geometrical entities".
-Giuseppe Peano

Here we will solve a problem from Principia of the translation of the kinematics of celestial objects into a fundamental equation of their dynamical interaction described by differential equations of the form: $\ddot{r} = f(r, \dot{r})$ where f is a vector-valued function (force divided by a scalar mass constant m) and r is a Euclidean vector with $\dot{r}$ being its tangent vector, as generally in classical mechanics forces can be functions of both position and velocity, as are magnetic and drag forces for instance.
The goal is to derive the function $f(r, \dot{r})$ from the kinematics of a given trajectory, and in expressing geometry through algebra directly,

We derive the law governing the gravitational interaction (gravitational inverse square law) of celestial bodies up to a constant from 2 (empirical) kinematical assumptions :

(1.) Orbits of celestial objects are conic sections, where the Sun is a focus.

(2.) The "angular momentum" concerning that focus point is time-symmetric. The second assumption is equivalent to that of the area swept out by the line joining the Sun and a celestial object being constant in time.

We will need to use the fact that the trajectory is a conic as per assumption (1.). It should be a function of position r and velocity $\dot{r}$, and also of the area so we can use the area sweeping time symmetry.
The basic combination of position and velocity that produces the area up to a scale factor is their outer product $r \wedge \dot{r}$. If we divide it by its magnitude (scale) we get the unit area bi-vector $i = r \hat{\wedge} \dot{r}$ whose square will give us a form of a conic with a constant real in the numerator and a vector dot product in the denominator; $\frac{c}{e \cdot \hat{d} + 1} = \frac{c}{e\cos(\theta) + 1} = |d|$, provided the eccentricity vector (Lagrange vector) e is constant in time.

$$
\begin{aligned}
i^2 = -1 &= \frac{r \wedge \dot{r}}{|\dot{r} \wedge r|}^2 = -\frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{(r \wedge \dot{r})(\dot{r} \wedge r)} && \text{from (7),(8)} \\
&= -\frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{(r \cdot (\dot{r} \cdot (\dot{r} \wedge r))} = -\frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{|r|\hat{r} \cdot (\dot{r}\dot{r} \wedge r)} && \text{from (2),(6)} \\
\Rightarrow |r| &= \frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{\hat{r} \cdot (\dot{r}\dot{r} \wedge r + k\hat{r} - k\hat{r})} && \text{-Assumption (1.) - The curve is a conic.} \\
&= \frac{(r \wedge \dot{r})(\dot{r} \wedge r)/k}{\hat{r} \cdot e + 1} && \text{- Thus } \dot{r}\dot{r} \wedge r - k\hat{r} \equiv ke \text{ must be a constant.} \\
\Rightarrow & \frac{d}{dt}(\dot{r}\dot{r} \wedge r - k\hat{r}) = 0 \\
&= r \wedge \dot{r}\ddot{r} - k\dot{\hat{r}} && \text{-Assumption (2.) - Conservation of angular momentum.} \\
&= |r|^2 \dot{\hat{r}} \wedge \hat{r}\ddot{r} - k\dot{\hat{r}} = 0 && \text{- As } \dot{\hat{r}} \cdot \dot{\hat{\boldsymbol{r}}} = \frac{1}{2}\hat{r} \,\dot{\cdot}\, \hat{\boldsymbol{r}} = 0. \\
\Rightarrow & -|r|^2 \dot{\hat{r}}\hat{r}\ddot{r} = k\dot{\hat{r}} \\
\Rightarrow & \ddot{r} = -k\frac{\hat{r}}{|r|^2}
\end{aligned}
$$

Additionally, with the knowledge of the orbital period for closed orbits (Kepler's 3rd law), the exact value of the constant k (mass m of the orbiting object is also suppressed in k here)can be determined.

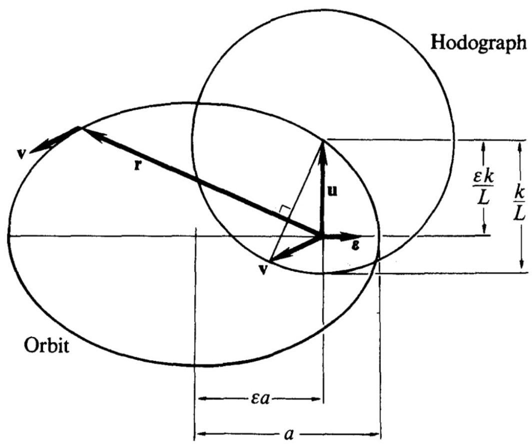

Figure 4: *Kepler trajectory [1]*

4 Symmetry of Euclidean space

The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures. In some way, Euclid's geometry must be deeply connected with the existence of the spin representation.
Hermann Weyl

The algebra of 3-d Euclidean vector space is 8-dimensional and contains the vector space as its first-grade subset. The 2-vectors and 3-vectors are the algebras 2^{nd}-grade 3-dimensional and 3^{rd}-grade 1-dimensional subsets respectively. They both are real imaginary numbers leaving vectors as the only real Euclidean algebra subspace whose elements have positive squares. This explains the appearance of imaginary numbers in so many areas of mathematics and physics as they are revealed to be a natural structure of Euclidean geometry. Still, it is quite interesting that simple Euclidean space contains this "complex "structure. We will see in the following that this is a special case of the spin structure of Euclidean space and that the imaginary number i is just a special kind of spinor or quaternion.
As complex numbers form a 2-dimensional vector space, so do real imaginary numbers represented by 2-vectors together with reals form a 4-d space of real quaternions, hence the name.

A reflection operator U(x,y) is a linear operator in its first slot of which the component parallel to the second slot input is reversed in the output. If we fix the second slot to be a constant vector u we get a linear operator U(x) reflecting x in the plane normal to a vector u :

U(x)=$-u^{-1}xu = (-u^{-1}\cdot x - u^{-1}\wedge x)u = -x\cdot -u^{-1}u - (u^{-1}\wedge x)\cdot u = x - 2x\cdot u^{-1}u = x - 2x\cdot \hat{u}\hat{u}$ -From (3b)

$= x - 2x_{\parallel} = x_{\perp} = x_{\parallel}$

This can also be seen as :

$-u^{-1}xu = -u(x_{\parallel} + x_{\perp})u = -u(x\cdot uu + x\wedge uu)u$ - from (3a),(3b)

$= -x\cdot uu + x\wedge uu$ -From (1), (2)

$= -x_{\parallel} + x_{\perp}$

-The scalar part of the induced transformation of a reflection operator U(x)on the geometric product computes the effect a reflection operator U(x) has on the inner product :

$(Ux)(Uy) = (Ux)\cdot(Uy) + (Ux)\wedge(Uy)$

$= (-uxu)(-uyu) = uxyu = ux\cdot yu + ux\wedge yu$

$(Ux)\cdot(Uy) = x\cdot y$

-Thus reflections are inner product-preserving operators called orthogonal transformations.

The determinant (action on the volume form) of reflections is the scalar which multiplies pseudoscalars as the induced transformation on the geometric product :

$(Ux)(Uy)(Uz) = (-uxu)(-uyu)(-uzu) = -u(xyz)u$

$-ux\wedge y\wedge zu = -x\wedge y\wedge z$ - Taking the trivector part , using (1).

Thus reflections aren't exterior product preserving (outer morphisms) as they don't preserve oriented n-volumes of n-spaces, which get multiplied by - 1 designating the value of their determinant.

Compositions of reflection operators determine their (linear) algebra :
$Vx = -vxv$

$VUx = vuxuv$

Reflection operators compose(multiply) through the fundamental multiplication of algebra, inherited from their direct vector (sandwich)representations.

ψ=VU - The composition of two reflections is a linear operator ψ ;

$\psi x = \hat{S}^{\dagger} x \hat{S}$

-Where $\tilde{x}$ is the reverse operation on multivectors which reverses the order of multiplication, here equivalent to (quaternion) conjugation, $\hat{S}$ is a unit quaternion also called rotor (unitary spinor in 3-d)

$\hat{S} = uv = u \cdot v + u \wedge v = e^{(1/2)i\theta}$ -From (9).

-Parametrisation by the multivector exponential gives explicit (bi-vector) angle relations(parametrization).

$\hat{S}^{\dagger} = \hat{S}^{-1} = uv = u \cdot v - u \wedge v = e^{(-1/2)i\theta}$

$\hat{S}\hat{S}^{\dagger} = 1$ -Here we use unitary spinors to get (special) orthogonal transformations and not dilations.

$i = \dfrac{u \wedge v}{|u \wedge v|}$ -The unit 2-blade i determines the rotation plane and is itself a$(\pi/2)$rotation operator in its plane (ch. 6.)eq.(7).

$x_{\parallel}\theta = x \cdot \theta\theta^{-1}\theta = x \cdot \theta = -\theta \cdot x = -\theta x_{\parallel}$ From (3c),(2).

$x_{\perp}\theta = x \wedge \theta\theta^{-1}\theta = x \wedge \theta = \theta \wedge x = \theta x_{\parallel}$ From (3d),(1).

$\hat{S}^{\dagger} x_{\parallel} = x_{\parallel} S$ -As in addition to the 2-blade θ, $\hat{S}$ has only a generally commuting (scalar)part which doesn't affect the sign.

$\hat{S}^{\dagger} x_{\perp} = x_{\perp} \hat{S}^{\dagger}$

$\hat{S}^{\dagger} x \hat{S} = \hat{S}^{\dagger}(x_{\perp} + x_{\parallel})\hat{S}$

$= \hat{S}^{\dagger}\hat{S} x_{\perp} + \hat{S}^2 x_{\parallel} = x_{\perp} + e^{i\theta} x_{\parallel}$

Reflections act on the parallel component ($x_{\parallel}$) (to the operation generator) by reversing its sign, while rotations rotate it (by a radian angle parameter).
Rotations are thus shown to be special(oriented volume-preserving) orthogonal transformations made of atomic orthogonal transformations called reflections directly represented by vectors.

The orthogonal group O(3) composed of inner product preserving linear transformations(rotations and reflections) is the fundamental, canonical algebraic structure of Euclidean space and is a subgroup of the group of general linear transformations GL(n) (of on the Euclidean vector space). O(n) together with translations form the group of isometries (distance preserving transformations) of Euclidean space called the Euclidean group E(n).

From (1) and (2) we see that a real quaternion can be expressed sum of a scalar (real) and a 2-vector (real Euclidean imaginary number).
Quaternions are numbers that dilate and rotate 3-Euclidean space. Unitary quaternions rotate 3-space. They are numbers expressing the (rotational) symmetry of Euclidean space.

Quaternions are a special case of more general (transformational) numbers called spinors. Euclidean spinors are elements of the even subalgebra of geometric algebra. They are the numbers that dilate and rotate n-dimensional (pseudo) Euclidean spaces. For a 3-d Euclidean space unitary spinors are (bi)rotations of Euclidean space forming the Spin (3) group with their composition (group product) simply being the fundamental (geometric) multiplication. Spin(3) determines a double cover of the special orthogonal group SO(3), (because opposite by-sign spinors are mapped to the same rotation due to two sandwiching minus signs).

Unitary Spinors of the Euclidean 3-space(real unit quaternions) form a lie group Spin(3) as they are naturally diffeomorphic to the 3-sphere. This unveils a very interesting connection (Hopf map)[11] between the 2- and 3-dimensional spheres and Euclidean geometry.

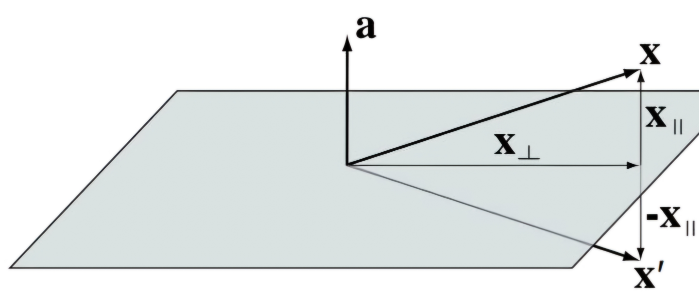

Figure 5: *Reflection in a plane represented by its unit normal vector [12]*

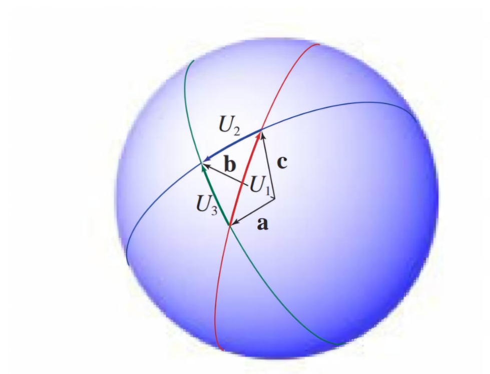

Figure 6: *The composition of 3D rotations is described algebraically by the product of spinors (here designated by the letter U) and represented geometrically by the addition of directed arcs on the unit 2-sphere with vectors a,b,c originating at the center of the sphere [12]*

5 Spinor dynamics

No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the 'square root' of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors.

- Michael Francis Atiyah

In the last chapter, we have seen that the rotation-dilation operator can be implemented using a spinor S which acts on the left and right side of a vector r:

$$r = S\hat{r}_0 S^\dagger \tag{5.1}$$

Where $\hat{r}_0$ is an arbitrary initial position vector direction. The spinor S here does not need to be unitary, then the absolute value of the spinor will give the scale of the position vector(unitary spinors only rotate vectors while general spinors also dilate them). Thus from the spinor S the position vector r can be determined uniquely. The converse does not hold as the position vector determines the position spinor only up to a "gauge transformation" $S \to GS$ (5.2), where G is a s spinor which leaves r invariant. Such a spinor is just the rotation operator with $\hat{r}_0$ as an eigenvector. It can be written explicitly as $e^{I\hat{r}_0\theta/2}$ (5.3) which rotates components of vectors in the plane dual to $\hat{r}_0$ by θ (I here is the unit pseudoscalar). This is analogous to the gauge transformation of the quantum spinor wave function. It can be said then that the equation (5.1) is invariant under the one-parameter group of gauge transformations determined by (5.2) and (5.3). The form (5.1) is particularly useful for describing the orientation of a rigid body with a corresponding set of principal axes e_k :

$$e_k = Se_kS^\dagger \tag{5.4}$$

To relate this to dynamics it is important to find a way to take derivatives of spinors. Though it will be used here only implicitly, the spinor time derivative is important in rigid body dynamics and geometric approaches to quantum mechanics. The proof follows the derivation in [1].

The derivative of a unitary spinor S is given by the expression

$$\dot{S} = \frac{1}{2}S\Omega = \frac{1}{2}Si\omega \tag{5.5}$$

$\Omega = I\omega$ is the bivector dual of the angular velocity vector ω. Since Ω is a bi-vector, from the properties of the reversion operator described in chapter 2. ;

$$\Omega^\dagger = -\Omega = -I\omega. \tag{5.6}$$

From the properties of reversion and ordinary derivatives it easily follows that

$$\dot{S}^\dagger = \frac{d}{dt}(S^\dagger) = \left(\frac{dS}{dt}\right)^\dagger. \tag{5.7}$$

From (5.5) using reversion the derivative of $S^\dagger$ can be computed ;

$$\dot{S}^\dagger = -\frac{1}{2}\Omega S^\dagger = -\frac{1}{2}i\omega S^\dagger. \tag{5.8}$$

Since S is unitary ;

$$S^\dagger S = 1. \tag{5.9}$$

With this 5.5 can be solved for

$$\Omega = 2S^\dagger\dot{S}. \tag{5.10}$$

If this is used to define Ω, it only remains to prove that Ω is a bivector. Differentiating (5.9) yields $\dot{S}^\dagger S + S^\dagger\dot{S} = 0$. From(5.7), $\Omega = 2S^\dagger\dot{S} = -2\dot{S}^\dagger S = -(2S^\dagger\dot{S})^\dagger = -\Omega^\dagger$, proving(5.6). The property $\Omega^\dagger = -\Omega$ ensures that Ω does not have scalar or vector parts, so it can only contain bivector and pseudoscalar parts. Furthermore, from the fact that the spinor S is even follows that $\dot{S}$ and $\dot{S}^\dagger$ are also even (multivectors) and by (5.7), Ω is even, so it does not have a pseudoscalar part. With this, we can conclude that Ω is a bivector which completes the proof for the spinor derivative.
With this (5.4) becomes :

$$\partial_t e_k = \omega \times e_k$$

This can be used to solve the Euler equation directly for a spherically symmetric body ($e_1 = e_2$) [1], which in the interest of brevity will not be further discussed here. The main aim of this section is to show that spinors and Clifford algebras are not limited descriptions to quantum or classical spin. This brings back into focus the fundamental problem of classical mechanics in formulating and solving equations of motion.
In 2-d the spinor position equation becomes : $r = S^2\hat{r}_0$ which can be written in Eulerian form :

$$r = \hat{r}_0|r|\exp(i\theta)$$

The "complex number" S now contains all of the dynamics and for the Kepler problem the diff. equation for motion for S is much simpler than that for r. Assuming that the plane of the angular momentum L is given by i, and with

$$|r| = SS^\dagger.$$

$$\dot{r} = 2\dot{S}S\hat{r}_0,$$

and by multiplying by $r_0 S^\dagger$ on the right :

$$2|r|\dot{S} = \dot{r}\hat{r}_0 S^\dagger = \dot{r}S\hat{r}_0.$$

The new variable s is defined by

$$\frac{d}{ds} = |r|\frac{d}{dt}, \quad \frac{dt}{ds} = |r|.$$

This yields ;

$$2\frac{dS}{ds} = \dot{r}S\hat{r}_0$$

and

$$2\frac{d^2S}{ds^2} = r\ddot{r}S\hat{r}_0 + \dot{r}\frac{dS}{ds}\hat{r}_0 = S\left(\ddot{r} + \frac{1}{2}\dot{r}^2\right).$$

With an inverse-square central force:

$$\mu\ddot{r} = -\frac{kr}{|r|^3}.$$

The spinor equation becomes ;

$$\frac{d^2S}{ds^2} = \frac{1}{2\mu}S\left(\frac{1}{2}\mu\dot{r}^2 - \frac{k}{|r|}\right) = \frac{E}{2\mu}S,$$

Which is the equation of the harmonic oscillator advantageous for being linear with and easily solvable.

$$\omega^2 = -\frac{E}{2\mu},$$

With the general solution being ;

$$S = Fe^{i\omega s} + Ge^{-i\omega s},$$

Where F and G are constants determined by initial conditions. Whereas in position space the particle flows along an ellipse with the origin being at the focus, the trajectory in "spinor space" is an ellipse but the origin is at its center and one full cycle in spinor space corresponds to two full orbits in position space. The linear nature of the spinor equation of motion makes it better suited for perturbation theory, and the absence of singularity at $r = 0$ provides better numerical stability(regularization)[1]. Additionally, the solutions are universal.
holding both for $E > 0$ and $E < 0$. When $E > 0$ trigonometric functions are simply replaced by exponentials. The universality of the equation is important in perturbation theory to capture both bound and unbound orbits simultaneously as one can be perturbed to the others.
Though the angular momentum bivector i fixes the Kepler problem to two dimensions, the spinor equation still works in 3-d.

$$r = S\hat{r}_0 S^\dagger,$$

S Now has four degrees of freedom (as it is a scalar plus a bivector in 3-d) while the position vector r only has 3 so we can use gauge freedom to simplify calculations.

$$|r| = SS^\dagger = S^\dagger S.$$

Differentiating ; $\dot{r}$:

$$\dot{r} = \dot{S}\hat{r}_0 S^\dagger + S\hat{r}_0\dot{S}^\dagger.$$

It would be convenient if this was equal to: $2\dot{S}\hat{r}_0 S^\dagger$ so that we have the same form as in 2-d. To that end, S can be fixed so that.

$$\dot{S}\hat{r}_0 S^\dagger - S\hat{r}_0\dot{S}^\dagger = \dot{S}\hat{r}_0 S^\dagger - (\dot{S}\hat{r}_0 S^\dagger)^\dagger = 0.$$

From (5.4) it then follows that $\dot{S}\hat{r}_0 S^\dagger$ contains only odd grade elements (grade-1 and grade-3). If its reverse is subtracted all that is left is the pseudoscalar part. Therefore S can be chosen such that.

$$\langle \dot{S}\hat{r}_0 S^\dagger \rangle_3 = 0,$$

With this we have

$$2\frac{dS}{ds} = \dot{r}S\hat{r}_0$$

and again differentiating ;

$$2\frac{d^2S}{ds^2} = (\ddot{r} + \frac{1}{2}\dot{r}^2)S.$$

For a central inverse-square force, the spinor equation of motion again takes the same form of a harmonic oscillator equation and with an additional perturbing force we finally have

$$2\mu\frac{d^2S}{ds^2} - ES = frS = |r|fS\hat{r}_0.$$

This derivation closely follows the one that can be found in [9]. The final expression can be handled using standard techniques from perturbation theory. The spinor equation above equation was first discovered (in matrix form) by Kustaanheimo and Stiefel[10] in 1964. Further refinement and analysis can be found in [1].

6 Quantum connection

"I learned to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting interesting mathematics."
— Paul Dirac

Experiments of Stern and Gerlach suggest that wave functions containing information about electrons should contain information about their spatial orientations. We have seen that a powerful way to do this is through the use of spinors which will be convenient here also for the reason that wave functions act on observables through "sandwich" products. When formulated through Clifford algebra the expectation value of an observable, in this case, the spin vector, can be written[8] as follows

$$s = \frac{1}{2}\hbar\psi\sigma_3\psi^{\dagger}.$$

The quantum mechanical expectation value expression[9] now reduces to

$$\langle\psi|\hat{s}_k|\psi\rangle = \frac{1}{2}\hbar(\sigma_k\psi\sigma_3\psi^{\dagger}) = \sigma_k \cdot s$$

This provides a new possible interpretation of quantum theory when it comes to spin measurement. Instead of the standard computation of the expectation value of a quantum mechanical operator, the measurement apparatus can be seen as acting as a spin polariser analogous to a photon polariser[9] with expression above calculating the expectation value of spin reduces to taking the projection of the kth component of the spin vector.
The square of the spinor wave function is

$$\rho = \psi\psi^{\dagger}.$$

So the quantum wave function ψ can be factored into

$$\psi = \rho^{1/2}\hat{S},$$

with $S = \rho^{-1/2}\psi$ where the spinor S is unitary ;$\hat{S}\hat{S}^{\dagger} = 1$. In this approach, Pauli algebra can be seen as isomorphic to the 3-d space algebra of unnormalized spinors[9]. The spin vector s then becomes ;

$$s = \frac{1}{2}\hbar\rho\hat{S}\sigma_3\hat{S}^{\dagger},$$

The double-sided action of the spinor wave function on the quantum observables is similar to how spinors act on classical observables. In 3 dimensions spinor functions must act on both sides for the spatial algebra to work, but with a half-angle parameter because they act twice, once on each side. As discussed in the previous chapter, an important observable in rigid body dynamics is the orientation of the body relative to some initial reference frame. Though the electron does not have rigid body structure, it can be seen as an "axially symmetric body" with a symmetry axis in the direction of the spin vector s being in the direction σ_3[11]. This is analogous to specifying a particular quantization axis in standard quantum mechanics formulations. The unitary spinor ψ rotates the initial reference frame into the frame $\{e_k\}$ and all physical vectors, like the spin vector s and other observables transform with this frame. Gauge phase transformations of ψ generate rotations in the plane determined by $e_1 e_2$ which along with the phase cannot be directly observed, however, the spinor wave function and its complex phase have an interesting and important "spinorial" transformation law that can now be explained geometrically.

If the spin vector s is rotated to another vector $S_\theta s S_\theta{}^\dagger$ the spinor ψ transforms according to

$$\psi \mapsto S_\theta \psi.$$

If S is written as

$$S_\theta = \exp(i\theta/2),$$

The wave function becomes

$$\psi' = S_\theta \psi = e^{i\theta/2}\psi.$$

A rotation of 2π brings all observables back to initial orientations but due to the spin representation the wave function only changes by a half phase so

$$\psi' = e^{i\pi}\psi = (\cos(\pi) - i\sin(\pi))\psi = -\psi.$$

Thus while all vectors and observables remain unchanged under a 2π rotation, the spinor wave function changes sign. After a 4π rotation all observables also remain unchanged but only then does the wave function return to its initial state.

To expound on the role of spinors in QM, we can see from chapter 5. that if S is unitary than its derivative is

$$\partial_t S = -\frac{1}{2} i\omega S,$$

so and from (5.4)

$$\partial_t e_k = \omega \times e_k,$$

Which is the kinematic equation for spin precession

$$\partial_t s = \omega \times s.$$

With this, an interesting equation for the energy can be derived [8] ;

$$E = \omega \cdot s = \frac{1}{2}\omega \cdot 2s.$$

This is equivalent to the classical rotational kinetic energy of a rigid body with 'spin' angular momentum $2s$[1] which may suggest another, more classical interpretation of the dynamics of electron 'motion' described by the spinor S, differing conceptually from the standard view involving the superposition of two quantum spin up and spin down states suggested by Pauli theory.

It can also be shown[8] that $\frac{i}{2}\hbar$ is the eigenvalue of the spin operator S acting on the wave function ψ: $S\psi = \frac{i}{2}\hbar\psi$.

The most surprising thing about the energy expression above is that it applies to any solution of the Schrodinger equation[8], where $\omega \times s = 0$. $e_1 \wedge e_2 = i$ represent the plane dual to the spin axis with the corresponding angular velocity ω, where the energy is associated with the rate of rotation (spin).

Conclusions

A simple bare Euclidean space contains hidden spin structure, as also its core symmetry space SO(3), is no longer a simple simply connected space, of which the spin(3) group of unitary spinors(quaternions) is a simply connected double cover and the fundamental manifestly invariant algebraic expression.
Euclidean spinors arise naturally as elements of the even subalgebra of Clifford algebra of the Euclidean vector space which defines the affine Euclidean space through group actions (geometric transformations) on its points. Euclidean vectors naturally generate their intrinsic algebra if the more general non-symmetric vector product is adopted instead of the special case of the symmetric inner product. Quantum wave functions are represented by spinors because spinors also act double-sidedly on variables and contain information about orientation in space. Furthermore, classical dynamics can also be formulated using spinors which have essentially the same geometric role as in quantum mechanics and bring many advantages over conventional formulations of rigid body mechanics and classical perturbation theory as the spinor equation of motion is linear and regular for the inverse square force with universal solutions. This leads to new interpretations of standard quantum mechanical frameworks and suggests new possibilities for more fundamental theories.

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