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Supervisor: prof. dr. sc. Predrag Dominis Prester

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Higher-spin-like symmetries and gauge models

Abstract

Higher derivative generalizations of translation symmetries (i.e. higher-spin-like symmetries) are utilized in this work in a novel approach to gauging, leading to a Yang-Mills-like theory defined over a symplectic manifold dubbed "master space". The developed theory incorporates the starting symmetries by using the Moyal product, has a weakly non-local action functional, it is perturbatively stable and admits a description in terms of an L_∞ algebra. The spectrum of the theory is analyzed in terms of Wigner's classification and to that purpose a novel unitary representation of the Lorentz group is built on the space of Hermite functions. The formulated field is massless and contains arbitrarily high helicities, while the square of the Pauli-Lubanski vector does not necessarily vanish, indicating that the model contains continuous spin field degrees of freedom.

The master space and the discovered symmetry can serve to build additional gauge field models, and we explicitly provide such candidate theories. In the Yang-Mills-like model we find an additional tower of conserved currents. Further, we display how matter fields can be modeled in the master space with coupling to the Yang-Mills like model and calculate scattering amplitudes for the simplest processes. Finally, we turn our attention to the low-spin sector of the theory and find a geometric description reminiscent of teleparallelism, with the induced linear connection related to Weitzenböck's. We apply the low-spin results to find additional general background solutions of the complete theory.

Keywords: higher spin fields, higher spin symmetry, gauge symmetry, non-commutative geometry

Simetrije višeg spina i baždarni modeli

Prošireni sažetak

Iako su slobodna polja višeg spina (gdje je $s > 2$) veoma dobro teorijski opisana, trenutno ne postoji potpuna interagirajuća teorija takvih polja u ravnom prostor-vremenu. Polja "nižeg spina" izvrsno služe opisu prirode; u slučaju spina $s = 1$ omogućuju opis elektromagnetske, slabe i jake sile dok u slučaju spina $s = 2$ sudjeluju u opisu gravitacije. Osim same znatiželje kakvu bi ulogu polja višeg spina mogla imati u opisu prirode, postoje brojne naznake kako bi njihova korisnost mogla biti velika u stvaranju konzistentnog opisa kvantne gravitacije.

Izravna ideja za formulaciju interagirajuće teorije viših spinova je početi sa slobodnim teorijama i potom ih deformirati red po red u deformacijskom parametru, pritom zahtijevajući konzistentnost s očekivanim simetrijama. Iako jasna, ta je ideja veoma teška u provedbi, pa literatura trenutno oskudijeva potpunim rezultatima, dok djelomični rezultati postoje. Tehnička zahtjevnost ovog problema nije jedina teškoća na putu ka interagirajućoj teoriji - postoji značajan broj teorema koji pod strogo određenim uvjetima zabranjuju postojanje interagirajuće teorije viših spinova. Te je uvjete važno uzeti u obzir, ne kao opstrukciju, već kao uputu kakva svojstva u teoriji očekujemo, kako bi interagirajuća teorija mogla postojati. Tri su najvažnija takva svojstva kojima možemo zaobići uvjete navedenih teorema; spektar teorije sadržava neograničen broj polja, u formulaciji teorije postoji određena vrsta ne-lokalnosti, a Lorentz-kovarijantnost nije ograničena na konačno dimenzionalne reprezentacije Lorentzove grupe. Ova se tri svojstva zaista pojavljuju u modelu koji razvijamo u ovom radu, prvenstveno baziranom na radovima [1, 2] i istraživanju započetom u [3].

Od trenutnih pristupa interagirajućim teorijama višeg spina ističemo Vasiljevljevu teoriju [4, 5] koja opisuje interagirajuća polja višeg spina u AdS prostor-vremenu, no bez mogućnosti prijelaza u ravno prostor-vrijeme i u formalizmu bez akcije, veoma udaljenom od konvencionalne teorije polja. U ravnom prostor-vremenu razvijena je "kiralna gravitacija višeg spina" [6, 7], no uz kompleksan Hamiltonijan i S-matricu jednaku identitetu.

Naš se pristup temelji na iskorištavanju simetrija jednostavnih modela materije, poput

masivnog kompleksnog skalarnog polja, koje je simetrično na transformacije

$$\delta_\varepsilon \phi(x) = \sum_{n=0}^{\infty} (-i)^{n+1} \varepsilon^{\mu_1 \dots \mu_n} \, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x) \, . \tag{1}$$

Kroz odgovarajuću reformulaciju modela materije, definiramo *master-prostor*, mnogostrukost koja je sačinjena od direktnog produkta prostor-vremena i pomoćnog prostora čije koordinate označavamo slovom u, nalik na fazni prostor točkaste čestice. Kroz korištenje Moyalovog $\star$ - produkta, polazne simetrije možemo izraziti kao

$$\delta_\varepsilon W_\phi(x,u) = i [W_\phi(x,u) \overset{\star}{,} \varepsilon(u)] \, , \tag{2}$$

gdje je $W_\phi(x,u) = \phi(x) \star \delta^d(u) \star \phi^\dagger(x)$ Wignerova funkcija sačinjena od polja materije. Na pozornici master prostora tada dolazimo do mogućnosti za formulaciju baždarnog polja $h_a(x,u)$ s infinitezimalnom baždarnom simetrijom

$$\delta_\varepsilon h_a(x,u) = \partial_a^x \varepsilon(x,u) + i \, [h_a(x,u) \overset{\star}{,} \varepsilon(x,u)] \tag{3}$$

i teorije formalno nalik Yang-Mills teoriji, a sa mogućnošću općenitije formulacije u kojoj je teorija nalik Yang-Mills-ovoj samo jedna moguća faza. Prema nedavnoj pretpostavci [8], konzistentne klasične teorije polja moguće je opisati L_∞ algebrom, pa tako pronalazimo opis i naše teorije kroz L_∞ algebru.

U spektru naše teorije se nalaze pobuđenja proizvoljno visokog heliciteta, a zbog neiščezavajuće vrijednosti kvartičnog Casimirovog operatora Poincaré-ove grupe, dolazimo do zaključka da naša teorija sadrži stupnjeve slobode beskonačnog spina - još uvijek nepotpuno istražene kategorije elementarnih čestica po Wignerovoj klasifikaciji. U svrhu analize spektra razvijena je i nova reprezentacija Lorentzove grupe na prostoru Hermitovih funkcija.

Nadalje, pokazujemo kako koristiti pronađenu simetriju za formulaciju dodatnih baždarnih modela, analizu mogućih simetrija i opis polja materije, koji primjenjujemo za izračun amplituda raspršenja jednostavnih procesa u granastoj aproksimaciji.

Konačno, posebnu pažnju posvećujemo dijelu naše teorije u kojem se nalaze samo pobuđenja niskog spina, pa pronalazimo specifičnu geometrijsku sliku, pomalo nalik teleparalelnoj geometriji, no uz linearnu koneksiju suprotnu Weitzenböckovoj.

Ključne riječi: polja višeg spina, simetrije višeg spina, baždarne simetrije, nekomutativna geometrija

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Chapter 1

Introduction and Motivation

Higher-spin theory, where higher-spin means $s > 2$, is a growing field of research whose origins coincide with the birth of quantum field theory. While free higher-spin field theories are well known, it is difficult to describe interactions. In this work we present a possible way forward. In the introductory chapter, we describe the motivation to pursue research in this direction, prepare the terminology, and introduce several established research programs. We then outline our approach, which departs from the traditional way of building an interacting higher-spin theory, and summarize the results presented in the main body of the thesis.

1.1 Motivation

It has been a long standing problem in theoretical physics to construct a satisfactory theory of quantum gravity. Conventional approaches with quantizing General Relativity result in a non-renormalizable theory, indicating insufficiency of General Relativity in describing high-energy processes. However, not even renormalizable theories are liberated from problems at high energies - the existence of a Landau pole in QED serves as an example. Today's experiments cannot probe the physics of our models at energies where these problems arise, so we turn instead to a detailed study of theoretical (mathematical and physical) constructs and aim to reach a self-consistent description of nature at all energy scales. In various approaches to solving these problems we encounter higher-spins.

Better behavior of scattering amplitudes at high energies was noticed in string theory; loop amplitudes are rendered finite which can be seen as due to an exchange of an infinite tower of higher-spin states [9]. Spin 2 being among them makes string theory a quantum theory of gravity. Sprung out of string theory is the conjectured AdS/CFT correspondence [10]. The conjecture is widely used and shown to work in numerous cases, evidence mounting towards its validity. The bulk side of the correspondence which should describe a quantum gravity theory routinely contains propagating degrees of freedom of higher-spin. Another motivation for studying higher-spins as a way towards quantum gravity comes from [11] where it was argued that one way to respect causality in higher derivative gravity is to extend the spectrum of the theory with higher-spin fields (see also [12] for a more constraining argument).

An important source of motivation is pure curiosity. Lower-spin fields are expedient in describing fundamental physics. What then is the role of higher-spin fields in our description of nature, and what is the appropriate theoretical framework for their formulation?

1.2 Higher-spin theory

An explanation of spin being a classifying number of elementary particles and their associated fields comes from Wigner's classification [13] where the particles are labeled by the eigenvalues of the Casimir invariants of the Poincaré algebra. Relevant cases include massive particles with definite spin (e.g. W and Z bosons), massless particles with definite helicity (e.g. photons), and a yet unobserved case of massless particles labeled by a dimensionful parameter, each containing an infinite number of helicities. Spin and helicity are unfortunately degenerate in the literature, and we will conform to this lore. A more detailed exposition of Wigner's classification will be given in chapter 4.

A direct way to ensure that a constructed field theory contains propagating degrees of freedom of a definite spin is to have the fields satisfy a certain set of partial differential equations, whose space of solutions carries the representation of the Poincaré group[1].

In the case of massive particles with integer-spin, a simple way to represent the Poincaré group is to use Lorentz tensor fields $\varphi_{\mu_1 \dots \mu_r}(x)$ satisfying the following differ-

[1]This is valid in case of a Minkowski background. In general one can choose another maximally symmetric space and represent its group of isometries. We will focus on the flat space.

ential and algebraic equations:

$$\left(\Box - m^2\right) \varphi_{\mu_1 \dots \mu_r}(x) = 0 \tag{1.1}$$

$$\partial^\nu \varphi_{\mu_1 \dots \nu \dots \mu_r}(x) = 0 \tag{1.2}$$

$$\varphi_{\mu_1 \dots \mu_r}(x) = \varphi_{(\mu_1 \dots \mu_r)}(x) \tag{1.3}$$

$$\eta^{\mu_1 \mu_2} \varphi_{\mu_1 \dots \mu_r}(x) = 0\,. \tag{1.4}$$

This set of equations ensures that the degrees of freedom contained inside a Lorentz tensor field of rank s are exclusively of spin s. In the massless case, things become more subtle. A possible solution is to keep the previous set of equations, albeit for $m = 0$, but recognize that manifest Lorentz covariance requires gauge invariance.

$$\varphi'_{\mu_1 \dots \mu_r}(x) = \varphi_{\mu_1 \dots \mu_r}(x) + \partial_{(\mu_1} \xi_{\mu_2 \dots \mu_{r-1})} \tag{1.5}$$

This means that a set of equations as above describes a single gauge choice, while the class of equivalence is larger. The construction outlined here uses tensors that are irreducible representations of the Lorentz group. Though straightforward, this is a very specific choice, and as we will see later, it is definitely not unique.

The description of integer linear (i.e. free) higher-spin fields on-shell was carried out in the early decades of the 20th century [14]. In 1947, it was completed by Bargmann and Wigner [15] for all integer and half-integer-spin fields. Once the equations were known on-shell, a Lagrangian description was sought after, additionally ignited by the discovery of composite higher-spin particles in particle accelerators in the 60's and 70's. The Lagrangian programme was completed by Singh and Hagen [16, 17] for massive particles in 1974, and by Fang and Fronsdal [18, 19] for massless particles in 1978 .

1.2.1 Interacting theories: a first view

Even though free higher-spin fields are well described in a Lagrangian formalism, a complete theory of their interactions is not known. We will focus our attention on the massless case. A minimal set of requirements for an interacting massless higher-spin theory would be:

- The linear part of the theory contains a description of propagation of at least one massless field with spin>2

- There exists an interacting part of the theory (with matter and possibly self-interacting)
- The theory has a stable vacuum

There is a straightforward way to describe how such a theory could be constructed, based on Gupta's idea of reconstructing General Relativity [20], formalized in [18] and named Noether's procedure. The advantage of this approach is having great control over the appearing degrees of freedom. One starts with an action $S_2[\phi]$ for a free field ϕ of spin s, with a field independent gauge symmetry[2]

$$\delta_\xi \phi = \partial \xi = \delta_0 \phi \,. \tag{1.6}$$

Deformations of the free action and the gauge symmetry with terms of higher order in the field can now be introduced

$$S[\phi] = S_2[\phi] + \epsilon S_3[\phi] + \epsilon^2 S_4[\phi]... \tag{1.7}$$

$$\delta_\xi \phi = \delta_0 \phi + \epsilon \delta_1 \phi + \epsilon^2 \delta_2 \phi + \dots \,. \tag{1.8}$$

Finally, one has to solve perturbatively the equation

$$\delta_\xi S = \epsilon(\delta_0 S_3 + \delta_1 S_2) + ... = 0 \,. \tag{1.9}$$

The antifield-BRST formalism [21, 22] is expedient for consistently posing and solving this deformation problem (see e.g. results in [23] and [24] for a review). Although this seems quite algorithmic, it is actually very complex, and has not yet yielded a complete result. For this reason, we will turn to an alternative way of learning about structures which should appear in an interacting theory. The technical severity of the problem is not the only difficulty on this path, and there are important lessons we can learn from the "no-go" theorems.

1.2.2 No-go theorems

There exist important results, based primarily on scattering amplitudes, that constrain the conditions for higher-spin particles to exist or interact. The name "no-go theorems" illustrates their strength; if their assumptions are satisfied, we cannot construct interacting higher-spin theories. It is important to emphasize that "no-go" does not mean

[2] Indices are suppressed in the following.

existence forbidden under any circumstances. Here we mention the most important of these theorems and draw attention to possible circumventions of their assumptions [25].

Weinberg's soft theorem [26] implies that in a scattering process with N external particles where a soft (low energy with respect to energy scales involved) massless spin s particle is also emitted, the external momenta must satisfy the following law:

$$\sum_{i=1}^{N} g_i^{(s)} p_i^{\mu_1} ... p_i^{\mu_{s-1}} = 0\,. \tag{1.10}$$

In the case of spin 1, the equation implies charge conservation, while in the case of spin 2 it implies universality of gravitational coupling. In the case of spin 3 or higher, there is no nontrivial solution to the above equation, implying that higher-spin particles cannot mediate long range forces. The theorem does not forbid short range interactions. In AdS space it does not apply since the S-matrix does not exist, and it is not clear how a degree of non-locality could affect the factorization properties of the soft amplitudes.

The Coleman-Mandula theorem [27] implies that a maximal amount of symmetry for a field theory is a semidirect sum of the Poincaré symmetries and internal symmetries such as ones in Yang-Mills theory. In the higher-spin symmetries, we find higher generalizations of translations, so they would have to be ruled out by this theorem. However, one of the crucial assumptions of this theorem is the existence of a finite number of particles under some mass-shell, a property routinely falsified in higher-spin theories which usually contain infinite towers of helicity states. Another possible way out comes from the fact that spacetime symmetries are here assumed to be encoded in a Poincaré algebra, however, one might allow for a more general extension of spacetime symmetries in the form of a universal enveloping algebra[3]. It is not yet completely clear how such an approach would work, but it would likely involve some sort of a bosonic extension of spacetime, similar to how supersymmetry has a fermionic extension of it [25].

The Weinberg-Witten theorem [28] states that any theory which has a Lorentz covariant and gauge invariant energy momentum tensor cannot carry spin higher than 1. This is already visible with the case of spin 2, where it is really impossible to construct a gauge invariant energy momentum tensor for the linear graviton field. Nevertheless, this does not forbid gravitons to self-interact, and the impossibility of a gauge invariant

[3] A universal enveloping algebra is a vector space of higher polynomials of Lie algebra generators with an inherited structure from the Lie bracket. In the literature, this is usually referred to as a "higher-spin algebra" where the starting Lie algebra is an isometry algebra of a maximally symmetric space.

energy momentum tensor is considered to reflect problems in localizing the energy of the gravitational field (see [29] for a generalization of the Weinberg-Witten theorem for theories without a gauge invariant energy-momentum tensor). The theorem also hinges on the assumption of higher-spin particles appearing in asymptotic states, which does not forbid a non-trivial interacting theory to exist.

To proceed with a possible construction of an interacting higher-spin theory, we take three important lessons from these theorems; the spectrum of the theory should have an infinite number of fields of different spin, there should be some degree of non-locality involved, and Lorentz covariance should not be bound to be achieved by conventional irreducible finite-dimensional tensor representations. As we will see below, in our models, these requests are not put in by hand, but are consequences of the construction.

1.2.3 Interacting theories - examples

Though the mentioned "no-go" theorems are severely constraining, positive examples exist (a good review can be found in [25, 30]). String theory is the most advanced theory containing states of higher-spin; in the perturbative spectrum of the theory, there appears a tower of excitations of ever increasing spin. In [31] a conjecture was put forward that a possible massless higher-spin theory might be an unbroken phase of string theory, whose states above spin 2 are massive. A breaking of higher-spin symmetries might be a mechanism by which the higher-spin particles obtain mass. String theory is thus an example of a mathematically consistent construction which predicts higher-spins.

One very important example of an interacting higher-spin theory in flat spacetime was created using the light-cone formalism, in which one works only with propagating degrees of freedom. This formulation enabled finding more interaction vertices than were known in the language of ordinary Lorentz tensors. The procedure starts with the formulation of a free theory, then one deforms it to the next polynomial level, and demands that the generators of the Poincaré algebra still satisfy the commutators as they did for the free theory. The solution is known as Chiral Higher-spin Gravity [6, 32]. The interesting aspect of this construction is a different number of positive and negative helicities (out of which stems the name "chiral"). Recently, in [7, 33] this theory was quantized and it was shown that the S-matrix is equal to unity, compatible with the no-go theorems. The Hamiltonian of Chiral Higher-spin Gravity is not Hermitian, indicating possible non-unitarity of the theory.

In AdS space, a self interacting higher-spin theory was formulated by Vasiliev in the late 80's [4, 5, 34]. By choosing a constantly curved background, it was possible to directly evade the S-matrix based no-go theorems. This construction, performed solely on-shell, starts by utilizing the higher-spin algebra, rather than Fronsdal fields, and promotes the symmetry to be local.

The Lie algebra of isometries of AdS space [35] is

$$i[P_m, P_n] = -\Lambda^2 M_{mn} \tag{1.11}$$

$$i[M_{mn}, P_r] = \eta_{nr} P_m - \eta_{mr} P_n \tag{1.12}$$

$$i[M_{mn}, M_{rs}] = \eta_{ms} M_{nr} - \eta_{mr} M_{ns} + \eta_{nr} M_{ms} - \eta_{ns} M_{mr} \,, \tag{1.13}$$

where M_{mn} are Lorentz generators and P_m are the AdS deformed translation generators, from which the higher-spin algebra $\mathfrak{hs}(4)$ is formed; an associative algebra of polynomials in the generators modulo a two sided ideal generated by the Lie algebra of isometries. This amounts to an algebra of polynomial operators in a certain ordering

$$\widehat{M}_{a_1 \cdots a_{s-1}, b_1 \cdots b_t} \sim (M_{ab})^t \, (P_a)^{s-1-t} \,. \tag{1.14}$$

Similar to Cartan's formulation of gravity, two fields are introduced which are valued in the higher-spin algebra, and full nonlinear equations are provided by the so called unfolded formalism.

There are some exact solutions known and it is true that the linearized part of the theory describes free higher-spin fields propagating on AdS space. It is also very useful in case of AdS/CFT correspondence, and recently twistors were explored as its possible geometric setting [36]. Apart from physical aspects, the mathematics of this theory are interesting in itself.

Nevertheless, Vasiliev's theory has some unattractive qualities. The language in which it is described is very far from the language of field theory, which makes it quite inaccessible. There is no action provided so it is not clear how to quantize it. It is formulated in AdS, without a regular limit to flat space (in simple terms, the theory contains higher derivatives weighted by the cosmological constant). It was originally thought to be local, but recently this has been seriously questioned [37], and although there are ongoing debates about the degree of non-locality, it is safe to say that a conventional notion of locality is not present.

1.3 Outline of the thesis

Differently from the examples described above, we will base our construction on a utilization of higher-spin-like symmetries of matter, which will be defined precisely below. In chapter 2, we will describe how a gauging procedure can be formulated in an extended manifold called the master space using the Moyal product. We will build a gauge theory analogous to Yang-Mills theory, state its properties and show how to formulate it in a more general way, manifestly displaying covariance with respect to the symmetry found through the gauging procedure. Following [8], we will show that the built model admits an L_∞ structure.

In chapter 3 we will construct a novel representation of the Lorentz group on the space of multi-dimensional Hermite functions. The results of chapter 3 are general and can be used directly even in contexts different from the work in this thesis, without reference to other chapters.

In chapter 4, we will analyze the spectrum of the theory for which the newly developed unitary representation of the Lorentz group will be used. It will be shown that the theory contains states of arbitrarily high helicity, but furthermore, that the master space formalism supports a description of infinite-spin particles.

In chapter 5 we will analyze the symmetries and conservation laws in our model, and put forward further candidate theories based on the discovered gauge symmetry. A connection to the matrix models will be highlighted.

In chapter 6 we will formulate a matter sector and describe possible ways of coupling matter to our gauge field. Tree-level scattering amplitudes for matter mediated by our gauge field will be calculated.

In chapter 7 we will focus on the low-spin ($s \leq 2$) sector of our theory and find that it induces a geometric picture with similarities to teleparallel geometry. This relation will be studied in more detail, and we will provide specific solutions to the field equations of the low-spin sector.

Finally, chapter 8 contains an overview of the presented material with comments on possible uses and future work.

The bulk of the thesis contains original work, apart from the small introductions on the L_∞ algebras in field theory, Wigner's classification and teleparallel geometry reported from existing literature. The appendices contain a review of mathematical tools used in calculations based on literature cited therein, as well as some details of the calculations in various chapters.

Chapters 2, 5, 6 and 7 are based on [1]:

- *"Gauging the Higher-Spin-Like Symmetries by the Moyal Product"*
  M. Cvitan, P. Dominis Prester, S. G. Giaccari, M. Paulišić, I. Vuković
  JHEP 06 (2021) p.144, arXiv: 2102.09254

Parts of chapter 4 are based on [2]:

- *"Gauging the Higher-Spin-Like Symmetries by the Moyal Product. II"*
  M. Cvitan, P. Dominis Prester, S. G. Giaccari, M. Paulišić, I. Vuković
  Symmetry 13.9 (2021) p.1581

Chapter 3, a major part of chapter 4 and section 7.3 contain work not yet published.

The first appearance of the gauge symmetry used in this work and a similar analysis of the L_∞ structure was in [3]:

- *"Worldline quantization of field theory, effective actions and L_∞ structure"*
  L. Bonora, M. Cvitan, P. Dominis Prester, S. G. Giaccari, M. Paulišić, T. Štemberga
  JHEP 04 (2018) p. 095, arXiv: 1802.02968

Chapter 2

Moyal Higher Spin Theory - origins and construction

In this chapter we describe how higher spin symmetries of matter fields can be utilized in a gauging procedure. To make the problem manageable we reformulate a sample model of matter in a new language by employing a Hilbert space description of matter and the Wigner-Weyl correspondence, and then formulate the Moyal Higher Spin (MHS) gauge field model on an extended manifold called the master space. We construct a Yang-Mills like theory using the newly developed concepts, show that it is classically perturbatively stable and display how the MHS structure allows for an even more general formulation. Finally, we show how the MHS Yang-Mills theory admits a description through the L_∞ algebra.

2.1 Higher-spin symmetries and the master space formulation

To make use of the gauging procedure we will examine global symmetries of a free complex massive scalar field, described by an action

$$S[\phi] = \frac{1}{2}\int d^dx \, \left(\partial_\mu\phi\partial^\mu\phi^\dagger - m^2\phi\phi^\dagger\right) \, . \tag{2.1}$$

Apart from the usual $U(1)$ symmetry, and the symmetry under translations, this action is symmetric under a whole tower of ever higher derivative symmetries which can be collected together in the following manner

$$\delta_\varepsilon \phi(x) = \sum_{n=0}^{\infty} (-i)^{n+1} \varepsilon^{\mu_1 \cdots \mu_n} \, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x) \, , \tag{2.2}$$

where $\varepsilon^{\mu_1 \cdots \mu_n}$ are completely symmetric constant Lorentz tensors, each of rank n. Focusing the attention on the particular choice $n = 0$ ($\delta_\varepsilon \phi(x) = -i\varepsilon\phi(x)$) we can recognize the infinitesimal form of the $U(1)$ gauge symmetry, while the choice $n = 1$ (($\delta_\varepsilon \phi(x) = -\varepsilon^\mu \partial_\mu \phi(x)$)) describes rigid spacetime translations. Higher rank cases are named higher spin symmetries, with rank n corresponding to spin $s = n + 1$. The significance of particular cases in (2.2) in constructing interaction terms between bosonic higher spin fields was first elucidated in [38], while the relation to the higher-spin algebra was explained in [39].

There is a well-known textbook trick (see e.g. [40]) to learn about the existence and gauge symmetry of a gauge field which consists in promoting a transformation parameter as in (2.2) to a function on spacetime (localizing or gauging the symmetry), while demanding that the symmetry of the action be preserved - the particular case of $n = 0$ leads to the Maxwell field. Analogous procedures can be done for the $n > 1$ cases[1]. There is, however, a strong argument that the gauging procedure is inconsistent if only a single localized higher-spin symmetry is used. Consider the commutator of variations generated by $n = 2\,(s = 3)$ parameters $\varepsilon_1^{\mu\nu}(x)$ and $\varepsilon_2^{\mu\nu}(x)$:

$$\begin{aligned} [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\phi = & \underbrace{\left(\varepsilon_2^{\alpha\beta} \partial_\alpha \partial_\beta \varepsilon_1^{\mu\nu} - \varepsilon_1^{\alpha\beta} \partial_\alpha \partial_\beta \varepsilon_2^{\mu\nu} \right)}_{s=3} \partial_\mu \partial_\nu \phi \\ & + \underbrace{2 \left(\varepsilon_2^{\mu\nu} \partial_\mu \varepsilon_1^{\alpha\beta} - \varepsilon_1^{\mu\nu} \partial_\mu \varepsilon_2^{\alpha\beta} \right)}_{s=4} \partial_\nu \partial_\alpha \partial_\beta \phi \end{aligned} \tag{2.3}$$

Even though the initial variations belonged to the case $s = 3$, the commutator contains $s = 4$ type transformations so the symmetry algebra is not closed. This problem is nonexistent only for the cases of $s = 1$ and $s = 2$, meaning that any consistent higher-spin construction necessarily needs to encompass the whole tower already written down in (2.2).

The gauging procedure usually starts by promoting the parameters $\varepsilon^{\mu_1 \cdots \mu_n}$ to functions on spacetime, but this direct approach might not be the best. To make this problem

[1] When $n > 1$ one speaks of true higher spin symmetries, while $n = 0, 1$ are usually referred to as low spin symmetries. We will refer to the whole tower 2.2 as higher spin even though the low spin cases are included.

tractable, we need to reformulate both the free scalar field action and the variations in a new language and consequently define the master space.

2.1.1 Hilbert space formulation

Any quadratic action such as (2.1) can formally be rewritten [41, 42] as a quadratic form

$$S[\phi] = \frac{1}{2} \langle\phi_r| \hat{K}_{rs} |\phi_s\rangle \ , \tag{2.4}$$

where the vectors $|\phi_r\rangle$ span a Hilbert space $\mathcal{H}$. The vectors are related to the field variables as $\phi_r(x) = \langle x|\phi_r\rangle$ where r can stand for any internal or Lorentz indices. With a complete set of operators acting on $\mathcal{H}$

$$[\hat{x}^a, \hat{u}_b] = i\delta^a_b \quad , \quad [\hat{x}^a, \hat{x}^b] = 0 = [\hat{u}_a, \hat{u}_b] \, , \tag{2.5}$$

we can build the (kinetic) operator

$$\hat{K} = \eta^{ab}\hat{u}_a\hat{u}_b - m^2 \, , \tag{2.6}$$

and prove by inserting a completeness relation $\int d^dx \, |x\rangle \langle x|$ that (2.4) is equal[2] to (2.1)

$$S[\phi] =\frac{1}{2}\int d^dx \, \langle\phi| \Big(|x\rangle \langle x| \, \eta^{ab}\hat{u}_a\hat{u}_b - m^2\Big) |\phi\rangle \tag{2.7}$$

$$=\frac{1}{2}\int d^dx \, \left(\partial_a\phi\partial^a\phi^\dagger - m^2\phi\phi^\dagger\right) \ . \tag{2.8}$$

It is now easy to manifestly display symmetries of the action (2.4) [41, 42, 43, 3]

$$S[\phi] = \frac{1}{2} \langle\phi_r| \hat{U}\hat{U}^{-1}\hat{K}_{rs}\hat{U}\hat{U}^{-1} |\phi_s\rangle \ . \tag{2.9}$$

With $\hat{U} = \exp(-i\hat{\mathcal{E}})$ and $\hat{\mathcal{E}}$ a hermitean operator we see that the linearized symmetry transformations are

$$\delta_\varepsilon |\phi\rangle = i\hat{\mathcal{E}} |\phi\rangle \, , \quad \delta_\varepsilon \langle\phi| = -i \langle\phi| \hat{\mathcal{E}} \tag{2.10}$$

$$\delta\hat{K}_{rs} = i[\hat{\mathcal{E}}, \hat{K}_{rs}] \, . \tag{2.11}$$

To reproduce the higher-spin transformations (2.2) we choose

$$\hat{\mathcal{E}}(\hat{u}) = \sum_{n=0}^{\infty} \varepsilon^{\mu_1\dots\mu_n}\hat{u}_{\mu_1}\dots\hat{u}_{\mu_n} \tag{2.12}$$

[2] Convenient rules are $\hat{u}_a |x\rangle \to i\partial^x_a |x\rangle \, , \langle x| \, \hat{u}_a \to -i\partial^x_a \langle x|$, being careful that these derivatives act on the full scalar product, i.e. $\langle x| \, \hat{u}_a\hat{u}_b \, |y\rangle = i(-i)\partial^x_a\partial^y_b\delta(x-y)$.

through which it is confirmed

$$\delta_\varepsilon \phi(x) = \langle x| \, i \, \hat{\mathcal{E}}(\hat{u}) \, |\phi\rangle \tag{2.13}$$

$$= \sum_{n=0}^{\infty} (-i)^{n+1} \varepsilon^{\mu_1 \dots \mu_n} \, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x) \, . \tag{2.14}$$

Note that (2.12) is a choice, while the symmetry transformation operator can be a general function $\hat{\mathcal{E}}(\hat{u})$ depending on operators $\hat{u}_\mu$.

2.1.2 Master space formulation

For the next step we will use the Wigner map and its inverse the Weyl map, which are defined and summarized in appendix A.2. We rewrite the matter action (2.4) as

$$S[\phi] = \frac{1}{2} \langle \phi| \, \hat{K} \, |\phi\rangle = \frac{1}{2} \mathrm{tr} \left(\hat{K} \, |\phi\rangle \, \langle \phi| \right) \, . \tag{2.15}$$

Since on the Hilbert space $\mathcal{H}$ we have defined a complete set of operators $\hat{x}^a, \hat{u}_b$, we can perform the Wigner map (A.21) on the action (2.15), which takes a Hilbert space operator and maps it to a function over a manifold resembling a phase space of a point particle, with coordinates x^a, u_b which we call the master space. A product of operators is by (A.27) mapped to a Moyal $\star$-product of functions, and the trace of an operator is mapped to an integral over the complete master space. For the kinetic operator we obtain

$$\int d^d q \, \langle x - \frac{q}{2} | \eta^{ab} \hat{u}_a \hat{u}_b - m^2 | x + \frac{q}{2} \rangle e^{iq\cdot u} = \eta^{ab} u_a u_b - m^2 \, , \tag{2.16}$$

while the projector $|\phi\rangle \, \langle \phi|$ is mapped to the Wigner function

$$\begin{aligned} \int d^d q \, \langle x - \frac{q}{2} \, |\phi\rangle \, \langle \phi| \, x + \frac{q}{2} \rangle \, e^{iq\cdot u} = & \int d^d q \, \phi(x - q/2) \phi^\dagger(x + q/2) \, e^{iq\cdot u} \\ =& (2\pi)^d \phi(x) \star \delta^d(u) \star \phi^\dagger(x) \, , \end{aligned} \tag{2.17}$$

where Moyal's star product $\star$ is defined by

$$a(x,u) \star b(x,u) = a(x,u) \, \exp\left[\frac{i}{2} \left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_u - \overrightarrow{\partial}_x \cdot \overleftarrow{\partial}_u \right) \right] b(x,u) \, . \tag{2.18}$$

The definition and properties of the Moyal product are given in appendix (A.1). The easiest way to prove that (2.17) is the Wigner function is by using the property (A.14) and to work backwards

$$
\begin{aligned}
(2\pi)^d \phi(x) \star \delta^d(u) \star \phi^\dagger(x) &= \int d^d q\, \phi(x) \star e^{iqu} \star \phi^\dagger(x) \\
&= \int d^d q\, \phi(x) \star \left[e^{iqu} \phi^\dagger(x + \frac{q}{2}) \right] \\
&= \int d^d q\, \phi(x - \frac{q}{2})\, e^{iqu} \phi^\dagger(x + \frac{q}{2}) \, . \qquad (2.19)
\end{aligned}
$$

Finally, the master-space form of the scalar field action is

$$
S[\phi] = \frac{1}{2} \int d^d x \, \frac{d^d u}{(2\pi)^d} \, (\eta^{ab} u_a u_b - m^2) \star W_\phi(x, u) \qquad (2.20)
$$

where x are spacetime coordinates and u are auxiliary coordinates of the same dimensionality. The master space is thus a non-commutative, $2d$-dimensional manifold where the non-commutativity is encoded in the Moyal commutators

$$
[x^a \overset{\star}{,} u_b] = i\delta^a_b, \quad [x^a \overset{\star}{,} x^b] = 0, \quad [u_a \overset{\star}{,} u_b] = 0 \, . \qquad (2.21)
$$

Note that the spacetime coordinates and auxiliary space coordinates remain commutative between themselves. Under Lorentz transformations, the coordinates transform as

$$
x^a \to \Lambda^a{}_b x^b, \qquad u_a \to \Lambda_a{}^b u_b \qquad (2.22)
$$

to keep the Moyal product Lorentz invariant.

The symmetry transformations (2.10) lead us to conclude that for the case of (2.12) the kinetic term is unchanged. The projector $|\phi\rangle \langle\phi|$ which maps to the Wigner function has the following transformation properties according to (2.10)

$$
\delta_\varepsilon \left(|\phi\rangle \langle\phi| \right) = i[\hat{\mathcal{E}}, |\phi\rangle \langle\phi|] \qquad (2.23)
$$

from which through the Wigner map we conclude that the Wigner function transforms as a Moyal commutator

$$
\delta_\varepsilon W_\phi(x, u) = i[W_\phi(x, u) \overset{\star}{,} \varepsilon(u)] \, . \qquad (2.24)
$$

At this point we additionally emphasize that $\varepsilon(u)$ is a general function of u, while a polynomial expansion similar to (2.12) could be made to make contact with a conventional approach to higher-spin symmetries. We will proceed in the more general point of view, but for completeness of the argument we note that a parameter $\varepsilon(u)$ of the form

$$
\varepsilon(u) = \sum_{n=0}^{\infty} \varepsilon^{\mu_1 \ldots \mu_n} \, u_{\mu_1} \ldots u_{\mu_n} \, . \qquad (2.25)
$$

reproduces the transformations (2.2, 2.14).

Gauging the symmetry amounts to promoting the symmetry parameters from rigid to local. In our setting, a rigid symmetry is described by a parameter depending solely on auxiliary coordinates u, while a local symmetry parameter has a dependence also on spacetime coordinates x. For simplicity, we set $m = 0$ in the matter action (2.20).

$$S = \frac{1}{2}\int d^dx\, \partial_\mu\phi\, \partial^\mu\phi^\dagger = \frac{1}{2}\int d^dx\, d^du\, u^a \star W_\phi \star u_a\ . \tag{2.26}$$

For a rigid parameter $\varepsilon(x,u) = \varepsilon(u)$ we indeed have a symmetry since[3]

$$\delta S = i\int d^dx\, d^du\, \frac{1}{2}([u^a \overset{\star}{,} W_\phi \star \varepsilon \star u_a] + [\varepsilon \overset{\star}{,} u^a] \star \{W_\phi \overset{\star}{,} u_a\})\ .$$

The first term is discarded as a Moyal commutator is a total derivative both in x and u coordinates, and the second term vanishes since $\varepsilon = \varepsilon(u)$, and a Moyal commutator between only auxiliary space variables vanishes.

In case where the gauge parameter $\varepsilon = \varepsilon(x,u)$ is also a function of x, the second term above does not vanish. If we demand that symmetry be preserved we must introduce a compensating vector field $h_a(x,u)$.

$$S = \frac{1}{2}\int d^dx\, d^du\, (u_a + h_a) \star W_\phi \star (u^a + h^a)\ . \tag{2.27}$$

The variation of the action becomes:

$$\delta S = \frac{1}{2}\int d^dx\, d^du\, \Big(\delta h_a \star \{W_\phi \overset{\star}{,} (u^a + h^a)\} + i(u_a + h_a) \star W_\phi \star \varepsilon \star (u^a + h^a) \tag{2.28}$$

$$- i(u_a + h_a) \star \varepsilon \star W_\phi \star (u^a + h^a)\Big)\ . \tag{2.29}$$

Following the logic above and discarding the boundary terms we obtain:

$$\delta S = \frac{1}{2}\int d^dx\, d^du\, (\delta h_a + i[\varepsilon \overset{\star}{,} (u_a + h_a)]) \star \{W_\phi \overset{\star}{,} (u^a + h^a)\}$$

from which we conclude that the action is locally invariant if the compensating field has the following infinitesimal transformation law:

$$\delta_\varepsilon h_a(x,u) = \partial_a \varepsilon(x,u) + i[h_a(x,u) \overset{\star}{,} \varepsilon(x,u)]\ . \tag{2.30}$$

This gauge symmetry made its first appearance in [3] as a symmetry of an action with a linear coupling of a tower of higher-spin fields to a Dirac fermion. A similar construction

[3] To see this add and subtract $+W_\phi \star \varepsilon \star u_a \star u^a - W_\phi \star \varepsilon \star u_a \star u^a + \varepsilon \star u_a \star W_\phi \star u^a - \varepsilon \star u_a \star W_\phi \star u^a$ and use cyclicity of the Moyal product under integration.

employing a linear coupling of a tower of higher spin fields to a massive scalar field was examined in [44, 43], whose results in the language of this chapter are presented in appendix B.1.

A most important feature of the master space construction is that the Lie algebra of symmetries is closed, as we can check

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] W_\phi(x,u) = i[\delta_{\varepsilon_1} W_\phi(x,u) \overset{\star}{,} \varepsilon_2(x,u)] - i[\delta_{\varepsilon_2} W_\phi(x,u) \overset{\star}{,} \varepsilon_1(x,u)] \tag{2.31}$$

$$= -[[W_\phi(x,u) \overset{\star}{,} \varepsilon_1(x,u)] \overset{\star}{,} \varepsilon_2(x,u)] + [[W_\phi(x,u) \overset{\star}{,} \varepsilon_2(x,u)] \overset{\star}{,} \varepsilon_1(x,u)] \tag{2.32}$$

$$= i[W_\phi(x,u) \overset{\star}{,} i[\varepsilon_1(x,u) \overset{\star}{,} \varepsilon_2(x,u)]] \,. \tag{2.33}$$

We conclude that the Lie algebra of (local) symmetries is non-abelian and infinite dimensional. The Lie bracket is provided by the Moyal commutator

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{i[\varepsilon_1 \overset{\star}{,} \varepsilon_2]} \,. \tag{2.34}$$

Through the reformulation of the matter model in the master space and the gauging procedure we have learned of the existence and gauge transformation properties of the master field $h_a(x,u)$. We take the master space as the stage on which we will build our model, with the algebra of functions realized with the Moyal product. We will call the symmetry transformations such as (2.24, 2.30) Moyal Higher Spin (MHS) transformations and define the master space fields according to possible transformation rules:

- Fundamental representation (infinitesimal parameter $\varepsilon(x,u)$)

$$\delta_\varepsilon \chi(x,u) \equiv -i\, \varepsilon(x,u) \star \chi(x,u) \tag{2.35}$$

- Fundamental representation (finite parameter $e_\star^{-i\,\mathcal{E}(x,u)}$)[4]

$$\phi^{\mathcal{E}}(x,u) = e_\star^{-i\,\mathcal{E}(x,u)} \star \phi(x,u) \tag{2.37}$$

- Adjoint representation (infinitesimal parameter $\varepsilon(x,u)$)

$$\delta_\varepsilon A(x,u) = -i\,[\varepsilon(x,u) \overset{\star}{,} A(x,u)] \tag{2.38}$$

[4]The $\star$-exponential function is naturally defined as

$$e_\star^{a(x,u)} = \sum_{n=0}^{\infty} \frac{1}{n!}\, a(x,u)^{\star n} \,. \tag{2.36}$$

More details can be found in appendix A.1

- Adjoint representation (finite parameter $e_\star^{-i\,\mathcal{E}(x,u)}$)

$$A^{\mathcal{E}}(x,u) = e_\star^{-i\,\mathcal{E}(x,u)} \star A(x,u) \star e_\star^{i\,\mathcal{E}(x,u)}\ . \tag{2.39}$$

A simple application of the Baker-Campbell-Hausdorff lemma guarantees that the large MHS transformations form a group, since it is always possible to find a solution $\overline{\mathcal{E}}(x,u)$ such that

$$e_\star^{-i\overline{\mathcal{E}}(x,u)} = e_\star^{-i\,\mathcal{E}_1(x,u)} \star e_\star^{-i\,\mathcal{E}_2(x,u)} \tag{2.40}$$

while the inverse is then obtained as

$$\left(e_\star^{-i\,\mathcal{E}(x,u)}\right)^{-1} = e_\star^{i\,\mathcal{E}(x,u)}\ . \tag{2.41}$$

2.2 MHS gauge potential and the Yang-Mills model

One way of discovering the master space gauge field $h_a(x,u)$ and its infinitesimal transformation properties was described in the previous section. Once we are equipped with the Moyal product and functions on the master space, we can independently follow the steps of the Yang Mills construction and generalize the previous conclusions. The conventional YM gauge field is a Lie Algebra valued 1-form $\mathbf{h}(x)$ defined on spacetime $\mathcal{M}$ which serves as a connection with transformation properties

$$\mathbf{h}^g(x) = g(x)\mathbf{h}(x)g(x)^{-1} - ig(x)\mathbf{d}g(x)^{-1} \tag{2.42}$$

with $\mathbf{d}$ the exterior derivative. In our case the group elements $g(x)$, which usually carry values of the Lie algebra, are functions on the master space, generating the MHS transformations. The same conclusion stands for the gauge field and we can define the connection transformation properties as

$$h_a^{\mathcal{E}}(x,u) \equiv e_\star^{-i\,\mathcal{E}(x,u)} \star h_a(x,u) \star e_\star^{i\,\mathcal{E}(x,u)} - ie_\star^{-i\,\mathcal{E}(x,u)} \star \partial_a^x e_\star^{i\,\mathcal{E}(x,u)}\ . \tag{2.43}$$

The definition above is compatible with the infinitesimal transformation properties found in the gauging procedure. If we expand each $e_\star^{-i\,\mathcal{E}(x,u)}$ with $\mathcal{E}(x,u) = \varepsilon(x,u)$ to emphasize linearization, and keep only linear terms in (2.43), it becomes

$$\delta_\varepsilon h_a(x,u) = \partial_a^x \varepsilon(x,u) + i\left[h_a(x,u) \overset{\star}{,} \varepsilon(x,u)\right] \tag{2.44}$$

identical to (2.30). We must keep in mind that the natural coordinates of the master space are x^a, u_b, thus, under Lorentz transformations, the MHS potential transforms as

$$h'_a(x', u') = \Lambda_a{}^b h_b(x, u) \tag{2.45}$$

where

$$x'^a = \Lambda^a{}_b x^b, \qquad u'_a = \Lambda_a{}^b u_b\,. \tag{2.46}$$

We can rewrite (2.45) as an active transformation and employ a matrix notation in the arguments with $(\Lambda^{-1} \cdot x)^a = (\Lambda^{-1})^a{}_b x^b$ and $(u \cdot \Lambda)_a = u_b \Lambda^b{}_a$ to obtain

$$h'_a(x, u) = \Lambda_a{}^b h_b(\Lambda^{-1} \cdot x, u \cdot \Lambda)\,. \tag{2.47}$$

The combination of the exterior derivative and the connection gives us the covariant derivative. The requirements for its definition are that (i) it is gradient linear, (ii) maps tensors into tensors, (iii) obeys the Leibniz rule and (iv) it is the inverse of the integral, which in our context for a covariant derivative denoted by $\mathcal{D}^\star_a$ means

$$\int d^d x\, d^d u\, \mathcal{D}^\star_a\, A^{a\dots}(x, u) = \text{(boundary terms)}\,. \tag{2.48}$$

This leads us to the definition

$$\mathcal{D}^\star_a \equiv \partial^x_a + i[h_a(x, u) \overset{\star}{,} \quad]\,. \tag{2.49}$$

The field strength (curvature) can now be defined as the covariant derivative of the gauge potential, which leads to

$$F_{ab}(x, u) = \partial^x_a h_b(x, u) - \partial^x_b h_a(x, u) + i\, [h_a(x, u) \overset{\star}{,} h_b(x, u)]\,. \tag{2.50}$$

Under MHS transformations we obtain expected transformation properties for infinitesimal transformations

$$\delta_\varepsilon F_{ab}(x, u) = i\, [F_{ab}(x, u) \overset{\star}{,} \varepsilon(x, u)]\,. \tag{2.51}$$

and equally expected for finite transformations

$$F^{\mathcal{E}}_{ab}(x, u) = e_\star^{-i\,\mathcal{E}(x,u)} \star F_{ab}(x, u) \star e_\star^{i\,\mathcal{E}(x,u)}\,. \tag{2.52}$$

The Bianchi identity follows directly:

$$\begin{aligned}
\mathcal{D}^\star_{[a} F_{bc]}(x, u) =& \partial_{[a}\partial_b h_{c]}(x, u) - \partial_{[a}\partial_c h_{b]}(x, u) \\
&+ i[\partial_{[a} h_b(x, u) \overset{\star}{,} h_{c]}(x, u)] + i[h_{[b}(x, u) \overset{\star}{,} \partial_a h_{c]}(x, u)] \\
&+ i[h_{[a}(x, u) \overset{\star}{,} \partial_b h_{c]}(x, u)] - i[h_{[a}(x, u) \overset{\star}{,} \partial_c h_{b]}(x, u)] \\
&- [h_{[a}(x, u) \overset{\star}{,} [h_b(x, u) \overset{\star}{,} h_{c]}(x, u)]] = 0\,,
\end{aligned} \tag{2.53}$$

where also the Jacobi identity for the Moyal product (A.11) was used.

An expected property of the field strength is that it measures the triviality of the configuration.

$$\mathbf{h} \text{ is pure gauge} \qquad \Longleftrightarrow \qquad \mathbf{F} = 0 \,. \tag{2.54}$$

This can be proven in the similar fashion as it is usually done in standard YM theory, using the fact that the Moyal product satisfies the algebraic properties of matrix multiplication. The proof of (2.54) is presented in Appendix B.3.

We now define the Moyal Higher Spin Yang-Mills (MHSYM) action as

$$\begin{aligned} S_{\text{ym}} &= -\frac{1}{4g_{\text{ym}}^2} \int d^d x \, d^d u \, F^{ab}(x,u) \star F_{ab}(x,u) \\ &= -\frac{1}{4g_{\text{ym}}^2} \int d^d x \, d^d u \, F^{ab}(x,u) F_{ab}(x,u) + (\text{boundary terms}) \,. \end{aligned} \tag{2.55}$$

The indices $a, b, ...$ are raised, lowered and contracted with the Minkowski metric η_{ab}. The presence of the Moyal star product, both explicitly and in the definition of the curvature tensor introduces higher derivatives into the action, making it weakly non-local[5]. There are at most quartic terms describing interactions. The boundary terms are irrelevant when equations of motion are sought for, but one has to be careful not to discard them when searching for conservation laws as we will examine in chapter 5. Due to the non-commutativity of the Moyal product, one should in general distinguish between the "left" and "right" functional derivatives of master space functionals

$$\delta_L A[h,\psi] = \int d^d x \, d^d u \, \delta h_a \star \frac{\delta_L A}{\delta h_a}, \qquad \delta_R A[h,\psi] = \int d^d x \, d^d u \, \frac{\delta_R A}{\delta h_a} \star \delta h_a \tag{2.56}$$

but on places where it does not make a difference, we will omit an explicit subscript, such as is the case when searching for equations of motion. For the MHSYM action the EoM for the master space field $h_a(x,u)$ are

$$\Box_x h_a - \partial_a^x \partial_b^x h^b + i\left(2[h^b \overset{\star}{,} \partial_b^x h_a] - [h_b \overset{\star}{,} \partial_a^x h^b] + [\partial_b^x h^b \overset{\star}{,} h_a]\right) + \left[h^b \overset{\star}{,} [h_a \overset{\star}{,} h_b]\right] = 0 \tag{2.57}$$

or written compactly

$$\mathcal{D}_a^\star F^{ba}(x,u) = 0 \,. \tag{2.58}$$

[5]A weak notion of non-locality entails higher derivatives appearing in the action in an unbounded order. A strong non-locality would entail operators such as $\frac{1}{\Box}$, and we do not encounter such operators in our theory.

Properties of the MHSYM model

In the next section, we will move to a more general construction of a gauge field theory in the master space. At this point, we emphasize some of the basic properties of the MHSYM model.

Apart from the MHS symmetry, the action (2.55) is invariant under the following transformations

$$h'_a(x',u') = \Lambda_a{}^b\, h_b(x,u) \quad , \quad x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + \xi^\mu \quad , \quad u'_\mu = \Lambda_\mu{}^\nu\, u_\nu + \tau_\mu \tag{2.59}$$

where Λ are Lorentz matrices and ξ^μ and τ_μ are arbitrary constant vectors. The Lorentz transformations acting in spacetime and the auxiliary space must be the same in order to keep the Moyal product invariant. We see that besides the standard Poincaré group (Λ, ξ) of spacetime isometries, there is also an independent group of translations in the auxiliary space. In chapter 6 we will show that a matter sector may break the auxiliary space translation symmetry.

The master field can be restricted to an odd function in the auxiliary space

$$h_a(x,-u) = -h_a(x,u)\ . \tag{2.60}$$

Indeed, it is easy to show that HS transformations are compatible with this if the MHS parameter is also restricted to be odd

$$\varepsilon(x,-u) = -\varepsilon(x,u)\ . \tag{2.61}$$

Using the expansion (2.25) it follows that this restriction corresponds to gauging only spin-even rigid HS transformations.

The dimension of the MHS potential is $(\text{length})^{-1}$ so the dimension of the coupling constant $g_{\rm ym}$ is $(\text{length})^{-2}$ for all d. It appears that the MHSYM theory has a scale, which we denote by ℓ_h, already at the classical level. However, we shall show in chapter 5 that the theory in the classical regime does not have an intrinsic scale and that the scale symmetry is spontaneously broken by the choice of the vacuum. To make contact with the canonical formalism it is natural to pass to the dimensionless auxiliary coordinates $\bar{u}$ and a rescaled coupling constant $\bar{g}_h$ and the MHS potential $\bar{h}_a$, defined by

$$\bar{u} = \ell_h u \qquad , \qquad \bar{g}_{\rm ym} = \ell_h^{d/2} g_{\rm ym} \qquad , \qquad \bar{h}_a = h_a/\bar{g}_{\rm ym}\ . \tag{2.62}$$

The dimension of $\bar{g}_{\rm ym}$ is $(\text{length})^{\frac{d}{2}-2}$, the same as in the standard Maxwell or Yang-Mills theories, and in $d=4$ it is zero. In the canonical normalization cubic terms and quartic terms in the action have the coupling given by $\bar{g}_{\rm ym}\ell_h^{D-1}$ and $\bar{g}_{\rm ym}^2\ell_h^D$, respectively, where D is the total number of spacetime derivatives in a given monomial.

An essential property of the MHSYM model is the stability of the vacuum solutions. Since $h_a(x,u)=0$ is a solution of (2.57) we can examine the behavior of the theory in the linear regime where we keep only quadratic terms in the action. To that end, the curvature tensors become

$$F_{ab}^{(2)} = \partial_a h_b(x,u) + \partial_b h_a(x,u) \tag{2.63}$$

and the linearized action is given by

$$S_{\rm ym}^{(2)} = -\frac{1}{4g_{\rm ym}^2}\int d^dx\, d^du\, F^{(2)ab}(x,u)F_{ab}^{(2)}(x,u)\,. \tag{2.64}$$

The action is formally similar to the one for the Maxwell theory, so we can immediately obtain the expression for the spatial energy density

$$U \approx \frac{1}{2g_{\rm ym}^2}\int d^{d-1}\mathbf{x}\int d^du\Big(\sum_j F_{0j}(x,u)^2 + \sum_{j<k} F_{jk}(x,u)^2\Big) \tag{2.65}$$

which is manifestly positive definite[6] and vanishes only for $h_a(x,u)=0$ (and gauge related configurations). We see that $h_a(x,u)=0$ is a perturbatively stable vacuum in the classical MHSYM theory. There are no runaway modes in the linearized regime, though one cannot exclude the possibility of their existence in the full theory, due to its non-locality and non-linearity. We have to impose on the MHS potential proper fall-off conditions at the boundary (infinity) of the auxiliary space to make sure that the action and observables such as energy and momentum are well-defined. This will be a decisive factor for determining the particle spectrum of the theory.

[6]The terms in the sum are ordinary squared, i.e. not by using the Minkowski metric. The energy density displays the same form as in Maxwell's theory $u=\frac{1}{2}\Big(\varepsilon_0\mathbf{E}^2+\frac{1}{\mu_0}\mathbf{B}^2\Big)$, albeit with an additional auxiliary space integration.

2.3 MHS Covariant formulation

2.3.1 MHS tensors

The MHS structure offers even a richer formalism than the one encountered in the Yang-Mills like formulation. We use a formal similarity with non-commutative field theories to borrow some of the techniques, and show that there exists a covariant frame-like formulation, important in understanding the emergent geometrical description described in chapter 7. As we will show, it also offers a better starting point for a background independent formulation and a direct relation to matrix models.

In the YM-like approach the basic object that covariantly transforms, in the adjoint representation, under MHS transformations is the MHS master field strength, see (2.52). Any master field $A(x, u)$ transforming in the same way, which is

$$A^{\mathcal{E}}_{ab\cdots}(x,u) = e_\star^{-i\,\mathcal{E}(x,u)} \star A_{ab\cdots}(x,u) \star e_\star^{i\,\mathcal{E}(x,u)} \tag{2.66}$$

we call an MHS tensor. In general it can have any number of Lorentz indices (denoted by Latin letters $a, b, \ldots$), on which MHS transformations do not act. For the moment we assume a flat background and trivial frames, so that Lorentz indices are raised and lowered with the Minkowski metric tensor. For infinitesimal MHS transformations this gives the MHS variation

$$\delta_\varepsilon A_{a\cdots}(x,u) = i\left[A_{a\cdots}(x,u) \stackrel{\star}{,} \varepsilon(x,u)\right] . \tag{2.67}$$

The important property of MHS tensors is that the Moyal product of MHS tensors is again an MHS tensor. This is a trivial consequence of (2.66) and (2.41).

$$A^{\mathcal{E}}(x,u) \star B^{\mathcal{E}}(x,u) = e_\star^{-i\,\mathcal{E}(x,u)} \star A(x,u) \star e_\star^{i\,\mathcal{E}(x,u)} \star e_\star^{-i\,\mathcal{E}(x,u)} \star B(x,u) \star e_\star^{i\,\mathcal{E}(x,u)} \tag{2.68}$$

$$= e_\star^{-i\,\mathcal{E}(x,u)} \star A(x,u) \star B(x,u) \star e_\star^{i\,\mathcal{E}(x,u)} \tag{2.69}$$

2.3.2 MHS vielbein

In standard YM gauge theories, to construct a covariant object from the gauge potential we need to take a derivative, which leads to constructing the gauge field strength. Non-commutativity of the MHS structure allows us to construct an MHS tensor without using derivatives, in the following way

$$e_a(x,u) \equiv u_a + h_a(x,u) . \tag{2.70}$$

Using (2.44) it is easy to show that $e_a(x,u)$ transforms under MHS variations as

$$\delta_\varepsilon e_a(x,u) = i\,[e_a(x,u) \stackrel{\star}{,} \varepsilon(x,u)]\ . \tag{2.71}$$

which is exactly the rule for the adjoint transformation law (2.67). The presence of such an object is not unexpected from the viewpoint of NC field theories (a closely related concept are covariant coordinates in non-commutative field theory [45], with an important difference being that in our case, spacetime remains commutative). By using $e_a(x,u)$ instead of the MHS potential $h_a(x,u)$ we can write all equations in the MHS gauge sector in a manifestly MHS covariant way (i.e., by using exclusively MHS tensors), a feat not possible in the standard YM theories.

We refer to $e_a(x,u)$ as the MHS vielbein. As we examine in more detail in chapters 4 and 7, it is sometimes useful to consider a Taylor expansion in the auxiliary coordinates

$$e_a(x,u) = \sum_{n=0}^{\infty} e_a^{(n)\mu_1\cdots\mu_n}(x)\,u_{\mu_1}\cdots u_{\mu_n}\ . \tag{2.72}$$

As with the higher spin symmetries (2.2), we call the n-th order term a spin $n{+}1$ spacetime component. If we perform a gauge transformation with a parameter containing only the $s=2$ contribution $\varepsilon(x,u)=\varepsilon^\mu u_\mu$, the spin-2 ($n=1$) spacetime component of (2.72) transforms under the MHS transformations as a vector frame under diffeomorphisms

$$\delta_\varepsilon {e_a}^{(1)\mu} = {e_a}^{(1)\nu}\partial_\nu\varepsilon^\mu - \varepsilon^\nu\partial_\nu {e_a}^{(1)\mu}\,. \tag{2.73}$$

When coupled to spacetime matter, this vector frame plays the role of the vielbein, as shown in chapter 6. This is the origin of the name MHS vielbein.

This expansion also illuminates the meaning of (2.70) and the preferred background it represents. Performing Taylor expansions (2.72) and equally

$$h_a(x,u) = \sum_{n=0}^{\infty} h_a^{(n)\mu_1\cdots\mu_n}(x)\,u_{\mu_1}\ldots u_{\mu_n}\ . \tag{2.74}$$

one obtains that the corresponding spacetime fields are connected through

$$e_a^{(n)\mu_1\ldots\mu_n}(x) = h_a^{(n)\mu_1\ldots\mu_n}(x) \qquad , \qquad n\neq 1 \tag{2.75}$$

and what corresponds to a vector frame is

$$e_a^{(1)\mu}(x) = {\delta_a}^\mu + h_a^{(1)\mu}(x)\ . \tag{2.76}$$

Through the vielbein interpretation we realize that (2.70) defines the MHS potential with respect to the empty Minkowski background

$$e_a = u_a \equiv \delta_a^\mu u_\mu \ . \tag{2.77}$$

This is not surprising, but it shows the limits of practical usability of (2.70). We can again see that the MHS vielbein is the fundamental object in the theory, and that (2.70) is sensible only if we are interested in expansions around the empty Minkowski vacuum.

Strictly speaking, to identify $e_a^{(1)\mu}(x)$ as a spacetime vielbein, an invertibility condition should be imposed. This condition is apparently not required in the MHS formalism, which opens the possibility of accommodating configurations and phases with non-geometric interpretations.

2.3.3 MHS covariant derivative and torsion

We have already seen in (2.49) that the covariant derivative is defined as

$$\mathcal{D}_a^\star = \partial_a^x + i\,[h_a(x,u) \overset{\star}{,} \quad] \tag{2.78}$$

and that it fulfills all the requirements for a covariant derivative when acting on an MHS tensor. Using (2.70) we obtain that the background independent formulation of (2.78) is given by

$$\mathcal{D}_a^\star = i\,[e_a(x,u) \overset{\star}{,} \quad] \ . \tag{2.79}$$

This form is not only more generic but also usually more convenient for performing calculations. Note that we can write the MHS variation of the MHS vielbein in a manifestly covariant form as

$$\delta_\varepsilon e_a(x,u) = \mathcal{D}_a^\star\, \varepsilon(x,u) \ . \tag{2.80}$$

Having defined the MHS vielbein and covariant derivative a natural object to construct is

$$T_{ab}(x,u) \equiv \mathcal{D}_a^\star\, e_b(x,u) = i\,[e_a(x,u) \overset{\star}{,} e_b(x,u)] \tag{2.81}$$

which is an antisymmetric MHS tensor

$$T_{ab}(x,u) = -T_{ba}(x,u) \ . \tag{2.82}$$

As we show in chapter 7, the Moyal bracket in the spin-2 sector behaves as the Lie bracket of vector fields. It then follows from (2.81) that T_{ab} can be interpreted both as the generalized anholonomy and the generalized torsion.[7] We shall refer to it as the MHS torsion. Expanding around the flat background (2.70), we get

$$T_{ab}(x,u) = \partial^x_a h_b(x,u) - \partial^x_b h_a(x,u) + i\,[h_a(x,u) \overset{\star}{,} h_b(x,u)] \tag{2.84}$$

which is the MHS field strength obtained in the YM-like construction and defined in (2.50). Our convention is to use the symbol T_{ab} in generic situations, and the symbol F_{ab} when (2.70) is meaningful.

A particularly interesting feature of the covariant MHS formulation is that there are no further independent MHS tensors which we could define motivated by a geometric analogy. In differential geometry the Riemann tensor is extracted from the commutator of covariant derivatives. The commutator of MHS covariant derivatives, acting on an arbitrary MHS tensor, gives

$$[\mathcal{D}^\star_a\,, \mathcal{D}^\star_b] A_{c\ldots}(x,u) = i\,[T_{ab}(x,u) \overset{\star}{,} A_{c\ldots}(x,u)] \tag{2.85}$$

thus it is defined by the MHS torsion. As a special case,

$$[\mathcal{D}^\star_a\,, \mathcal{D}^\star_b]\, e_c(x,u) = \mathcal{D}^\star_c T_{ba}(x,u) \tag{2.86}$$

from which we see that there is no extra independent structure in our formalism corresponding to the generalized Riemann tensor. Using the Jacobi identity (A.11) it is straightforward to show that the MHS torsion satisfies the MHS Bianchi identity,

$$\mathcal{D}^\star_a\, T_{bc}(x,u) + \mathcal{D}^\star_b\, T_{ca}(x,u) + \mathcal{D}^\star_c\, T_{ab}(x,u) = 0 \tag{2.87}$$

which is the same as (2.53).

Note that by putting $A_{c\ldots}(x,u) = \varepsilon(x,u)$ in (2.85) we can write the MHS variation of the MHS torsion as

$$\delta_\varepsilon T_{ab}(x,u) = [\mathcal{D}^\star_a\,, \mathcal{D}^\star_b]\ \varepsilon(x,u) \tag{2.88}$$

$$= \mathcal{D}^\star_a\, \delta_\varepsilon e_b(x,u) - \mathcal{D}^\star_b\, \delta_\varepsilon e_a(x,u)\ . \tag{2.89}$$

[7] The latter is obvious when we write T_{ab} in the following form

$$T_{ab} = \mathcal{D}^\star_a\, e_b(x,u) - \mathcal{D}^\star_b\, e_a(x,u) - i\,[e_a(x,u) \overset{\star}{,} e_b(x,u)]\ . \tag{2.83}$$

2.3.4 The MHS metric

The simplest HS tensor without frame indices in our formalism is

$$g(x,u) \equiv e_a(x,u) \star e^a(x,u)\ . \tag{2.90}$$

For the obvious reason we call it the MHS metric. If we use (2.72) and a similar Taylor expansion for the MHS metric

$$g(x,u) = \sum_{s=0}^{\infty} g_{(s)}^{\mu_1\ldots\mu_s}(x)\, u_{\mu_1}\cdots u_{\mu_s} \tag{2.91}$$

it follows from (2.90) that the $s=2$ component is given by

$$g_{(2)}^{\mu\nu}(x) = \eta^{ab} e_a^{(1)\mu}(x)\, e_b^{(1)\nu}(x) + (\text{HS contributions}) \tag{2.92}$$

where every monomial in "(HS contributions)" contains field(s) $e_a^{(n)\mu_1\ldots\mu_n}(x)$ with $n \geq 2$, which is spin ≥ 3. Up to spin $s>2$ contributions, this is exactly the relation between a metric and a vielbein in standard differential geometry.

If we expand the MHS vielbein as in (2.70), then the natural way to expand the HS metric is

$$g(x,u) \equiv u^2 + h(x,u)\ . \tag{2.93}$$

Taylor expanding both sides around $u=0$, we get for $s \neq 2$

$$g_{(s)}^{\mu_1\ldots\mu_s}(x) = h_{(s)}^{\mu_1\ldots\mu_s}(x) \qquad , \qquad s \neq 2 \tag{2.94}$$

and for $s=2$

$$g_{(2)}^{\mu\nu}(x) = \eta^{\mu\nu} + h_{(2)}^{\mu\nu}(x)\ . \tag{2.95}$$

We see that the MHS field $h(x,u)$ measures the deviation from the flat background. Using (2.90), (2.93) and (2.71) we get the MHS variation of $h(x,u)$

$$\delta_\varepsilon h(x,u) = 2(u\cdot\partial_x)\varepsilon(x,u) + i\,[h(x,u) \stackrel{\star}{,} \varepsilon(x,u)] \tag{2.96}$$

which is exactly the variation found in [44, 43] in the analysis of MHS symmetries of the free Klein-Gordon field linearly coupled to the infinite tower of spacetime HS fields. In [44, 43] it was argued that $h(x,u)$ should be a composite field, and here we made it explicit. In chapter 6 we show that $h(x,u)$ indeed is the field which couples minimally to the Klein-Gordon field in the MHS formalism. Using (2.90), (2.93) and (2.70) we obtain

$$h(x,u) = 2\,u^a h_a(x,u) + h_a(x,u) \star h^a(x,u)\ . \tag{2.97}$$

In particular the $s = 0$ component of $h(x, u)$, which provides seagull vertices for $s \geq 1$ interactions when coupled to a Klein-Gordon field, is

$$h_{(0)}(x) = h_a(x, u) \star h^a(x, u)\Big|_{u=0} . \tag{2.98}$$

The MHS covariant derivative is not metric-compatible since

$$Q_a(x, u) \equiv \mathcal{D}^{\star}_a g(x, u) = i \left[e_a(x, u) \overset{\star}{,} g(x, u)\right] \tag{2.99}$$

is generally not vanishing. We refer to the HS tensor $Q_a(x, u)$ as the MHS nonmetricity tensor. The underlying geometry in our construction appears not to be of the Riemann-Cartan type. Note that the MHS nonmetricity tensor (2.99) can be written as

$$Q_a(x, u) = \{e_b(x, u) \overset{\star}{,} T_a{}^b(x, u)\} \tag{2.100}$$

i.e., it is completely determined by the MHS torsion.

To summarize, the geometry emerging in the MHS theory has all fundamental tensors (torsion, Riemann tensor and nonmetricity) non-vanishing. While the geometry may look exotic at a first glance, it is in fact closely related to the teleparallel geometry. In chapter 7 we study in detail the spin-2 sector, and show that the emergent linear connection is opposite[8] to the Weitzenböck connection.

2.4 L_∞ structure of the MHS model

It was conjectured in [8] that any classical, perturbatively defined field theory can be described by an appropriate L_∞ algebra, a vast generalization of Lie algebras, which we will soon define. This is an exciting prospect, which gives hope into identifying the crucial recipes for field theory, abstracting away from any specific ingredients. Further merit to this conjecture can be found through the dual relationship of L_∞ algebras and the Batalin-Vilkovisky formalism [47]. Examples are plentiful and we refer the reader to the citation tree of [8, 48] and to the references in the contemporary theses [49, 50].

The MHS theory can also be described by an L_∞ algebra, which is one of the first results on the MHS symmetry obtained already in [3]. There, the analysis was based on the MHS gauge transformations and the symmetry of the effective action, which entails all theories symmetric under MHS transformations, while here, we will focus on the specific

[8]To use the terminology of [46].

instance of the Yang-Mills-like theory as defined above and show that it admits an L_∞ structure, both in the formulation through the MHS potential and the MHS vielbein.

The definition of the L_∞ algebra and our analysis will follow closely [8]. The history of its appearance in physics is summarized in [51].

2.4.1 Definition and properties of the L_∞ algebra

An introduction to L_∞ algebras in terms of generalizing Lie algebras is given in appendix A.3. Here we immediately state the definition of the L_∞ algebra precisely. An L_∞-algebra is a $\mathbb{Z}$-graded vector space X

$$X = \bigoplus_n X_n, \quad n \in \mathbb{Z} \tag{2.101}$$

where

$$\mathrm{x_k} \equiv \deg x_k = n \Rightarrow x_k \in X_n \tag{2.102}$$

together with a collection of multilinear products $\ell_1, \ell_2, \ell_3, ...$

$$\ell_k : \underbrace{X \times \cdots \times X}_{k} \rightarrow X \tag{2.103}$$

that are graded commutative with a degree

$$\deg \ell_k = k - 2 \tag{2.104}$$

meaning

$$\deg(\ell_k(x_1, \ldots, x_k)) = k - 2 + \sum_{i=1}^{k} \deg(x_i)\,. \tag{2.105}$$

The graded-commutativity is given by

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = (-1)^\sigma \epsilon(\sigma; x) \ell_k(x_1, \ldots, x_k) \tag{2.106}$$

where σ is a permutation of k labels, $(-1)^\sigma$ is negative (positive) if the permutation is odd (even), and $\epsilon(\sigma; x)$ is the Koszul sign. For example

$$\ell_2(x_2, x_1) = (-1)^{1+\deg(x_1)\deg(x_2)} \ell_2(x_1, x_2)\,. \tag{2.107}$$

We can use the notation $\deg(x_i) = \mathrm{x}_i$. One direct way to determine the Koszul sign is to consider a graded commutative algebra

$$x_i \wedge x_j = (-1)^{\mathrm{x}_i \mathrm{x}_j} x_j \wedge x_i \tag{2.108}$$

and find the Koszul sign $\epsilon(\sigma; x)$ through

$$x_1 \wedge \ldots \wedge x_k = \epsilon(\sigma; x) x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(k)} \,. \tag{2.109}$$

The products ℓ_k in L_∞ algebra satisfy a very specific quadratic identity enumerated by a positive integer n equal to the number of inputs. Schematically, the identities are of the form[9]

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \ell_j(\ell_i(...)) = 0 \tag{2.110}$$

and in full detail the identities are

$$\sum_{i+j=n+1} (-1)^{i(j-1)} (-1)^\sigma \epsilon(\sigma; x) \ell_j \left(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\right) = 0\,. \tag{2.111}$$

The sum is over the so-called "unshuffles", which are permutations that partially keep the order $\sigma(1) < ... < \sigma(i)$ and $\sigma(i+1) < ... < \sigma(n)$.

Let us unpack the first 3 identities which we will explicitly use below

n=1

$$\ell_1(\ell_1(x)) = 0 \tag{2.112}$$

n=2

$$\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-)^{\mathrm{x}_1} \ell_2(x_1, \ell_1(x_2)) \tag{2.113}$$

n=3

$$\begin{aligned} 0 =& \ell_1(\ell_3(x_1, x_2, x_3)) \\ &+ \ell_3(\ell_1(x_1), x_2, x_3) + (-)^{\mathrm{x}_1} \ell_3(x_1, \ell_1(x_2), x_3) + (-)^{(\mathrm{x}_1+\mathrm{x}_2)} \ell_3(x_1, x_2, \ell_1(x_3)) \\ &+ \ell_2(\ell_2(x_1, x_2), x_3) + (-)^{\mathrm{x}_1(\mathrm{x}_2+\mathrm{x}_3)} \ell_2(\ell_2(x_2, x_3), x_1) + (-)^{(\mathrm{x}_2+\mathrm{x}_1)\mathrm{x}_3} \ell_2(\ell_2(x_3, x_1), x_2) \end{aligned}$$

2.4.2 L_∞ algebras in field theory

The authors in [8] put forward a claim that all field theories which can be defined perturbatively are structured with respect to some L_∞ algebra. Based on earlier work in closed string field theory [52], they provide a general form of how equations of motion and gauge transformations should look when expressed through the L_∞ products ℓ_k. They

[9] Compare with (A.34, A.35)

also define an inner product of vectors and by using it identify the general form of the action expressed as an inner product of fields with some ℓ_k. The inner product is defined to satisfy

$$\langle x_1, x_2\rangle =(-)^{\mathrm{x}_1\mathrm{x}_2}\langle x_2, x_1\rangle \tag{2.114}$$

$$\langle x, \ell_n(x_1, \ldots, x_n)\rangle =(-)^{\mathrm{x}\mathrm{x}_1}\langle x_1, \ell_n(x, \ldots, x_n)\rangle \tag{2.115}$$

An L_∞ algebra together with the inner product is called cyclic.

By direct comparison of the general equations and the specific forms of gauge transformations, action, and field equations in a specific theory, it is possible to extract information on how exactly the products ℓ_k act, and confirm that these products really follow the L_∞ structure. This is what we will do below. It is assumed that there are no vector spaces with degree $d \leq -3$. The gauge parameters Λ, fields Ψ and equations of motion $\mathcal{F}$ belong to the vector spaces

$$\Lambda \in X_0, \quad \Psi \in X_{-1}, \quad \mathcal{F} \in X_{-2} \tag{2.116}$$

where the subscript indicates the degree of elements belonging to that vector space. A field theory is structured through an L_∞ algebra as follows

- Gauge transformations are

$$\delta_\Lambda \Psi =\ell_1(\Lambda) + \ell_2(\Lambda, \Psi) - \frac{1}{2}\ell_3(\Lambda, \Psi, \Psi) - \frac{1}{3!}\ell_4(\Lambda, \Psi, \Psi, \Psi) + \cdots \tag{2.117}$$

  or in general

$$\delta_\Lambda \Psi = \sum_{n=0}^{\infty} \frac{1}{n!}(-)^{\frac{n(n-1)}{2}}\ell_{n+1}(\Lambda, \Psi^n)\,. \tag{2.118}$$

- With (2.117) it can be seen that a commutator of gauge transformations is

$$\delta_{\Lambda_1}\delta_{\Lambda_2} - \delta_{\Lambda_2}\delta_{\Lambda_1} = \delta_{\Lambda_{12}} + \delta^T_{\Lambda_1,\Lambda_2} \tag{2.119}$$

  where $\delta_{\Lambda_{12}}$ is an ordinary gauge transformation with a parameter[10]

$$\Lambda_{12} = \ell_2(\Lambda_2, \Lambda_1) + \ell_3(\Lambda_2, \Lambda_1, \Psi) - \frac{1}{2}\ell_4(\Lambda_2, \Lambda_1, \Psi, \Psi, ...) \tag{2.120}$$

  and $\delta^T_{\Lambda_1,\Lambda_2}$ is an "equation-of-motion" type symmetry, a transformation which vanishes on-shell, stated explicitly in e.g. [53]

$$\delta^T_{\Lambda_1,\Lambda_2}\Psi = \sum_{n=0}^{\infty} \frac{(-)^{\frac{n(n-1)}{2}}}{n!}\ell_{n+3}(\Lambda_1, \Lambda_2, \mathcal{F}, \Psi, ..., \Psi)\,. \tag{2.121}$$

[10]Can be shown perturbatively through the use of L_∞ identities.

- The action is

$$S = \frac{1}{2}\langle \Psi, \ell_1(\Psi)\rangle - \frac{1}{3!}\langle \Psi, \ell_2(\Psi,\Psi)\rangle - \frac{1}{4!}\langle \Psi, \ell_3(\Psi,\Psi,\Psi)\rangle + \cdots \tag{2.122}$$

or in general

$$S = \sum_{n=1}^{\infty} \frac{1}{(n+1)!}(-)^{\frac{n(n-1)}{2}} \langle \Psi, \ell_n(\Psi^n)\rangle\,. \tag{2.123}$$

- Field equations are

$$\mathcal{F}(\Psi) = \ell_1(\Psi) - \frac{1}{2}\ell_2(\Psi,\Psi) - \frac{1}{3!}\ell_3(\Psi,\Psi,\Psi) + \cdots \tag{2.124}$$

or in general

$$\mathcal{F}(\Psi) = \sum_{n=1}^{\infty} \frac{1}{n!}(-)^{\frac{n(n-1)}{2}} \ell_n(\Psi^n)\,. \tag{2.125}$$

Without reference to a specific field theory, we can identify the products

$$\ell_k(\mathcal{F}, \underbrace{\Psi, \Psi, ..., \Psi}_{k-1}) = 0 \tag{2.126}$$

since their degree would be $(k-2) + (-2) + (k-1)\cdot(-1) = -3$, and a vector space with degree X_{-3} does not exist.

2.4.3 L_∞ structure of MHSYM theory: MHS potential formulation

First we display the L_∞ structure of the MHSYM model where we use $h_a(x,u)$ as the fundamental field. The steps in the analysis are analogous to the analysis of the ordinary Yang-Mills theory. We will present the final result and then prove it.

The graded vector space is formed by three spaces of interest of fixed degree

- X_0 contains gauge parameters $\varepsilon(x,u)$, $\deg\varepsilon = 0$.
- X_{-1} contains fields $h_a(x,u)$, $\deg h_a = -1$.
- X_{-2} contains equations of motion $\mathcal{F}_a(x,u)$, $\deg\mathcal{F}_a = -2$.

Spaces in other degrees are empty. We also define an inner product as an integral of the Moyal product of factors over the master space.

$$\langle A, B\rangle \equiv \int d^dx\, d^du\, A(x,u) \star B(x,u) \tag{2.127}$$

The only non vanishing products, with Lorentz indices stated, are given by

$$
\begin{aligned}
\ell_1(\varepsilon)_a =& \partial_a \varepsilon && (2.128)\\
\ell_1(h)_a =& (\Box \delta_a^b - \partial_a \partial^b) h_b && (2.129)\\
\ell_2(\varepsilon_1, \varepsilon_2) =& i[\varepsilon_2 \overset{\star}{,} \varepsilon_1] && (2.130)\\
\ell_2(\varepsilon, h)_a =& i[h_a \overset{\star}{,} \varepsilon] && (2.131)\\
\ell_2(h_1, h_2)_a =& -i\Big(\partial^b [h_{1b} \overset{\star}{,} h_{2a}] + \partial^b [h_{2b} \overset{\star}{,} h_{1a}] \\
& + [h^{1b} \overset{\star}{,} \partial_b h_{2a} - \partial_a h_{2b}] + [h^{2b} \overset{\star}{,} \partial_b h_{1a} - \partial_a h_{1b}]\Big) && (2.132)\\
\ell_2(\mathcal{F}, \varepsilon)_a =& i[\varepsilon \overset{\star}{,} \mathcal{F}_a] && (2.133)\\
\ell_3(h_1, h_2, h_3)_a =& -[h_1^b \overset{\star}{,} [h_{2a} \overset{\star}{,} h_{3b}]] - [h_2^b \overset{\star}{,} [h_{3a} \overset{\star}{,} h_{1b}]] - [h_3^b \overset{\star}{,} [h_{1a} \overset{\star}{,} h_{2b}]] \\
& - [h_2^b \overset{\star}{,} [h_{1a} \overset{\star}{,} h_{3b}]] - [h_1^b \overset{\star}{,} [h_{3a} \overset{\star}{,} h_{2b}]] - [h_3^b \overset{\star}{,} [h_{2a} \overset{\star}{,} h_{1b}]] && (2.134)
\end{aligned}
$$

Proof: identifying products

Most of the products can easily be determined by adapting the MHS expressions to (2.117, 2.124). There are a few more that are identified through the L_∞ identities. We start with the gauge transformations and compare them to the general expressions.

$$
\begin{aligned}
\delta_\varepsilon h_a =& \partial_a \varepsilon + i[h_a \overset{\star}{,} \varepsilon] && (2.135)\\
\delta_\varepsilon h_a =& \ell_1(\varepsilon)_a + \ell_2(\varepsilon, h)_a - \frac{1}{2}\ell_3(\varepsilon, h, h)_a - \frac{1}{3!}\ell_4(\varepsilon, h, h, h)_a + \cdots && (2.136)
\end{aligned}
$$

We identify

$$\ell_1(\varepsilon)_a = \partial_a \varepsilon, \quad \ell_2(\varepsilon, h)_a = i[h_a \overset{\star}{,} \varepsilon], \quad \ell_{k>2}(\varepsilon, h, ..., h)_a = 0\,. \tag{2.137}$$

Equations of motion give us

$$
\begin{aligned}
\mathcal{F}_a &= \Box_x h_a - \partial_a^x \partial_b^x h^b + i\left(\partial^b [h_b \overset{\star}{,} h_a] + [h^b \overset{\star}{,} \partial_b h_a - \partial_a h_b]\right) + \left[h^b \overset{\star}{,} [h_a \overset{\star}{,} h_b]\right] \\
\mathcal{F}_a &= \ell_1(h)_a - \frac{1}{2}\ell_2(h, h)_a - \frac{1}{3!}\ell_3(h, h, h)_a + \cdots && (2.138)
\end{aligned}
$$

meaning

$$
\begin{aligned}
&\ell_1(h)_a = (\Box \delta_a^b - \partial_a \partial^b) h_b, \quad \ell_2(h, h)_a = -2i(\partial^b [h_b \overset{\star}{,} h_a] + [h^b \overset{\star}{,} \partial_b h_a - \partial_a h_b]) \\
&\ell_3(h, h, h)_a = -3![h^b \overset{\star}{,} [h_a \overset{\star}{,} h_b]], \quad \ell_{k>3}(h, .., h) = 0\,. && (2.139)
\end{aligned}
$$

It is straightforward to check that the identified products furnish the correct expression for the action

$$
\begin{aligned}
S &= \frac{1}{2}\langle h, \ell_1(h)\rangle - \frac{1}{3!}\langle h, \ell_2(h,h)\rangle - \frac{1}{4!}\langle h, \ell_3(h,h,h)\rangle + \cdots \\
&= \int d^dx\, d^du \left(\frac{1}{2}h_a \star (\Box\eta^{ab} - \partial^a\partial^b)h_b - \partial_a h_b \star i[h^a \stackrel{\star}{,} h^b] - \frac{1}{4}[h_a \stackrel{\star}{,} h_b] \star [h^a \stackrel{\star}{,} h^b]\right)
\end{aligned}
$$

From the products (2.139), which are symmetric due to the fields being $\deg h_a = 1$, and multilinear by definition, we can extract the products with different field inputs. For instance, in the case of $n = 2$, we have the identity

$$
\ell_2(h_1 + h_2, h_1 + h_2) = 2\ell_2(h_1, h_2) + \ell_2(h_1, h_1) + \ell_2(h_2, h_2) \tag{2.140}
$$

so we find

$$
\begin{aligned}
\ell_2(h_1, h_2)_a =& -\frac{1}{2}\cdot 2i \left(\partial^b[h_{1b} + h_{2b} \stackrel{\star}{,} h_{1a} + h_{2a}] + [h_1^b + h_2^b \stackrel{\star}{,} \partial_b h_{1a} + \partial_b h_{2a} - \partial_a h_{1b} - \partial_a h_{2b}]\right) \\
& - \partial^b[h_{1b} \stackrel{\star}{,} h_{1a}] + [h_1^b \stackrel{\star}{,} \partial_b h_{1a} - \partial_a h_{1b}] - \partial^b[h_{2b} \stackrel{\star}{,} h_{2a}] - [h_2^b \stackrel{\star}{,} \partial_b h_{2a} - \partial_a h_{2b}] \\
=& -i\Big(\partial^b[h_{1b} \stackrel{\star}{,} h_{2a}] + \partial^b[h_{2b} \stackrel{\star}{,} h_{1a}] + [h_1^b \stackrel{\star}{,} \partial_b h_{2a} - \partial_a h_{2b}] + [h_1^b \stackrel{\star}{,} \partial_b h_{1a} - \partial_a h_{1b}]\Big)\,.
\end{aligned} \tag{2.141}
$$

In a similar manner, we can find the non-diagonal product

$$
\begin{aligned}
\ell_3(h_1, h_2, h_3)_a =& -[h_1^b \stackrel{\star}{,} [h_{2a} \stackrel{\star}{,} h_{3b}]] - [h_2^b \stackrel{\star}{,} [h_{3a} \stackrel{\star}{,} h_{1b}]] - [h_3^b \stackrel{\star}{,} [h_{1a} \stackrel{\star}{,} h_{2b}]] \\
& - [h_2^b \stackrel{\star}{,} [h_{1a} \stackrel{\star}{,} h_{3b}]] - [h_1^b \stackrel{\star}{,} [h_{3a} \stackrel{\star}{,} h_{2b}]] - [h_3^b \stackrel{\star}{,} [h_{2a} \stackrel{\star}{,} h_{1b}]]
\end{aligned} \tag{2.142}
$$

The gauge algebra in our case is given by

$$
\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]h_a =& \delta_{i[\varepsilon_1 \stackrel{\star}{,} \varepsilon_2]}h_a \\
=& \delta_{\varepsilon_1}(\ell_1(\varepsilon_2)_a + \ell_2(\varepsilon_2, h)_a) - \delta_{\varepsilon_2}(\ell_1(\varepsilon_1)_a + \ell_2(\varepsilon_1, h)_a) \\
=& \ell_2(\varepsilon_2, \delta_{\varepsilon_1}h)_a - \ell_2(\varepsilon_1, \delta_{\varepsilon_2}h)_a \\
=& \ell_2(\varepsilon_2, \ell_1(\varepsilon_1) + \ell_2(\varepsilon_1, h))_a - \ell_2(\varepsilon_1, \ell_1(\varepsilon_2) + \ell_2(\varepsilon_2, h))_a \\
=& -\ell_2(\ell_1(\varepsilon_1), \varepsilon_2)_a - \ell_2(\varepsilon_1, \ell_1(\varepsilon_2))_a + \ell_2(\varepsilon_2, \ell_2(\varepsilon_1, h))_a - \ell_2(\varepsilon_1, \ell_2(\varepsilon_2, h))_a
\end{aligned}
$$

To identify the L_∞ structure, we can use the $n = 2$ identity

$$
\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-)^{x_1}\ell_2(x_1, \ell_1(x_2)) \tag{2.143}
$$

and obtain

$$
\begin{aligned}
\delta_{i[\varepsilon_1 \stackrel{\star}{,} \varepsilon_2]}h_a =& \partial_a\left(i[\varepsilon_1 \stackrel{\star}{,} \varepsilon_2]\right) + i[h_a \stackrel{\star}{,} i[\varepsilon_1 \stackrel{\star}{,} \varepsilon_2]] \\
=& -\ell_1(\ell_2(\varepsilon_1, \varepsilon_2))_a + \ell_2(\varepsilon_2, \ell_2(\varepsilon_1, h))_a - \ell_2(\varepsilon_1, \ell_2(\varepsilon_2, h))_a\,.
\end{aligned} \tag{2.144}
$$

In a perturbative comparison in powers of h_a we find

$$-\ell_1(\ell_2(\varepsilon_1,\varepsilon_2))_a = \partial_a \left(i[\varepsilon_1 \overset{\star}{,} \varepsilon_2]\right) . \tag{2.145}$$

Since $\ell_2(\varepsilon_1,\varepsilon_2)$ is an element of degree 0, we can use the rule $\ell_1(\varepsilon)_a = \partial_a\varepsilon$ to conclude

$$\ell_2(\varepsilon_1,\varepsilon_2) = -i[\varepsilon_1 \overset{\star}{,} \varepsilon_2] \tag{2.146}$$

At the linear order in h_a, we can confirm the product rule (2.137) which now takes the from

$$\ell_2(\varepsilon_2,\ell_2(\varepsilon_1,h))_a = i[[h_a \overset{\star}{,} \varepsilon_1] \overset{\star}{,} \varepsilon_2] \tag{2.147}$$

The gauge commutator enables us also through (2.120) to identify

$$\ell_3(\varepsilon_1,\varepsilon_2,h) = 0, \quad \ell_{k>3}(\varepsilon_1,\varepsilon_2,h,...,h) = 0 \tag{2.148}$$

while the fact that our algebra closes off-shell implies through (2.121)

$$\ell_3(\varepsilon_1,\varepsilon_2,E) = 0 \tag{2.149}$$

Proof: verifying identities

We will organize the verification of (2.111) by the number of inputs.

n=1 The identity to be checked is

$$\ell_1(\ell_1(x)) = 0\,. \tag{2.150}$$

The only non-trivial identity to be checked is

$$\ell_1(\ell_1(\varepsilon))_a = \ell_1(\partial_a\varepsilon) = (\Box\eta_{ab} - \partial_a\partial_b)\partial^b\varepsilon = 0\,. \tag{2.151}$$

which is satisfied.

n=2 The relevant identity is

$$\ell_1(\ell_2(x_1,x_2)) = \ell_2(\ell_1(x_1),x_2) + (-)^{x_1}\ell_2(x_1,\ell_1(x_2))\,. \tag{2.152}$$

At input degree 0 we have

$$\ell_1(\ell_2(\varepsilon_1,\varepsilon_2)) = \ell_2(\ell_1(\varepsilon_1),\varepsilon_2) + (-)^{\varepsilon_1}\ell_2(\varepsilon_1,\ell_1(\varepsilon_2)) \tag{2.153}$$

$$-i\ell_1([\varepsilon_1 \overset{\star}{,} \varepsilon_2])_a = \ell_2(\partial_a\varepsilon_1,\varepsilon_2) + \ell_2(\varepsilon_1,\partial_a\varepsilon_2) \tag{2.154}$$

$$-i\partial_a[\varepsilon_1 \overset{\star}{,} \varepsilon_2] = \ell_2(\partial_a\varepsilon_1,\varepsilon_2) + \ell_2(\varepsilon_1,\partial_a\varepsilon_2)\,. \tag{2.155}$$

We can use the map with inputs of degree 1 and 0

$$\ell_2(h,\varepsilon)_a = i[\varepsilon \overset{\star}{,} h_a] \tag{2.156}$$

to obtain that the identity is valid

$$-i[\partial_a\varepsilon_1 \overset{\star}{,} \varepsilon_2] - i[\varepsilon_1 \overset{\star}{,} \partial_a\varepsilon_2] = i[\varepsilon_2 \overset{\star}{,} \partial_a\varepsilon_1] - i[\varepsilon_1 \overset{\star}{,} \partial_a\varepsilon_2]\,. \tag{2.157}$$

At input degree -1 we have

$$\ell_1(\ell_2(h,\varepsilon)) = \ell_2(\ell_1(h),\varepsilon)_a + (-)^{\mathrm{h}}\ell_2(h,\ell_1(\varepsilon)) \tag{2.158}$$

$$i\ell_1([\varepsilon \overset{\star}{,} h_a]) = \ell_2(\ell_1(h),\varepsilon)_a - \ell_2(h,\partial_a\varepsilon)\,. \tag{2.159}$$

Since we have not yet identified the product of the type $\ell_2(E,\varepsilon)$ where $E \in X_{-2}, \varepsilon \in X_0$, we explicitly calculate the rest, and recognize it through this identity. The non-symmetric version of ℓ_2 was reported above in (2.141), so we have

$$i(\Box\delta_a^b - \partial_a\partial^b)[\varepsilon \overset{\star}{,} h_b] = \ell_2(\ell_1(h),\varepsilon)_a + \tag{2.160}$$

$$i\Big(\partial^b[h_b \overset{\star}{,} \partial_a\varepsilon] + \partial^b[\partial_b\varepsilon \overset{\star}{,} h_a] + [h^b \overset{\star}{,} \partial_b\partial_a\varepsilon - \partial_a\partial_b\varepsilon] + [\partial^b\varepsilon \overset{\star}{,} \partial_b h_a - \partial_a h_b]\Big) \tag{2.161}$$

which can be reduced to

$$\ell_2(\ell_1(h),\varepsilon)_a = i[\varepsilon \overset{\star}{,} \Box h_a - \partial_a\partial^b h_b] \tag{2.162}$$

$$= i[\varepsilon \overset{\star}{,} \ell_1(h)_a]\,. \tag{2.163}$$

For that reason we define the product ℓ_2 with inputs $E \in X_{-2}$ and $\varepsilon \in X_0$ as

$$\ell_2(E,\varepsilon) = i[\varepsilon \overset{\star}{,} E] \tag{2.164}$$

At input degree -2, we have a trivial equality, since the target vector space would be of degree -3.

n=3 The relevant identity is

$$\begin{aligned} 0 =&\ell_1(\ell_3(x_1,x_2,x_3)) \\ &+\ell_3(\ell_1(x_1),x_2,x_3) + (-)^{\mathrm{x}_1}\ell_3(x_1,\ell_1(x_2),x_3) + (-)^{(\mathrm{x}_1+\mathrm{x}_2)}\ell_3(x_1,x_2,\ell_1(x_3)) \\ &+\ell_2(\ell_2(x_1,x_2),x_3) + (-)^{\mathrm{x}_1(\mathrm{x}_2+\mathrm{x}_3)}\ell_2(\ell_2(x_2,x_3),x_1) + (-)^{(\mathrm{x}_2+\mathrm{x}_1)\mathrm{x}_3}\ell_2(\ell_2(x_3,x_1),x_2)\,. \end{aligned} \tag{2.165}$$

At input degree 0 we have

$$
\begin{aligned}
0 =&\ell_1(\ell_3(\varepsilon_1,\varepsilon_2,\varepsilon_3))\\
&+\ell_3(\ell_1(\varepsilon_1),\varepsilon_2,\varepsilon_3)+\ell_3(\varepsilon_1,\ell_1(\varepsilon_2),\varepsilon_3)+\ell_3(\varepsilon_1,\varepsilon_2,\ell_1(\varepsilon_3))\\
&+\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),\varepsilon_3)+\ell_2(\ell_2(\varepsilon_2,\varepsilon_3),\varepsilon_1)+\ell_2(\ell_2(\varepsilon_3,\varepsilon_1),\varepsilon_2)\,.
\end{aligned}
\tag{2.166}
$$

Because of (2.148) the second line vanishes, and we find

$$
\begin{aligned}
\ell_1(\ell_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)) = &-\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),\varepsilon_3)-\ell_2(\ell_2(\varepsilon_2,\varepsilon_3),\varepsilon_1)-\ell_2(\ell_2(\varepsilon_3,\varepsilon_1),\varepsilon_2)\\
=&-\ell_2(i[\varepsilon_2\overset{\star}{,}\varepsilon_1],\varepsilon_3)-\ell_2(i[\varepsilon_3\overset{\star}{,}\varepsilon_2],\varepsilon_1)-\ell_2(i[\varepsilon_1\overset{\star}{,}\varepsilon_3],\varepsilon_2)\\
=&[\varepsilon_3\overset{\star}{,}[\varepsilon_2\overset{\star}{,}\varepsilon_1]]+[\varepsilon_1\overset{\star}{,}[\varepsilon_3\overset{\star}{,}\varepsilon_2]]+[\varepsilon_2\overset{\star}{,}[\varepsilon_1\overset{\star}{,}\varepsilon_3]]\\
=&0
\end{aligned}
\tag{2.167}
$$

We can thus confirm that it is consistent to set $\ell_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)=0$.
At input degree -1 we have

$$
\begin{aligned}
0 =&\ell_1(\ell_3(\varepsilon_1,\varepsilon_2,h))_a\\
&+\ell_3(\ell_1(\varepsilon_1),\varepsilon_2,h)_a+(-)^{\varepsilon_1}\ell_3(\varepsilon_1,\ell_1(\varepsilon_2),h)_a+(-)^{\varepsilon_1+\varepsilon_2}\ell_3(\varepsilon_1,\varepsilon_2,\ell_1(h))_a\\
&+\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),h)_a+(-)^{\varepsilon_1(\varepsilon_2+\mathrm{h})}\ell_2(\ell_2(\varepsilon_2,h),\varepsilon_1)_a+(-)^{(\varepsilon_2+\varepsilon_1)\mathrm{h}}\ell_2(\ell_2(h,\varepsilon_1),\varepsilon_2)_a\,.
\end{aligned}
\tag{2.168}
$$

The first line vanishes due to (2.148), the second line vanishes due to (2.137). We are left with the third line

$$
\begin{aligned}
0 =&\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),h)_a+\ell_2(\ell_2(\varepsilon_2,h),\varepsilon_1)_a+\ell_2(\ell_2(h,\varepsilon_1),\varepsilon_2)_a\\
=&\ell_2(i[\varepsilon_2\overset{\star}{,}\varepsilon_1],h)_a+\ell_2(i[h\overset{\star}{,}\varepsilon_2],\varepsilon_1)_a+\ell_2(i[\varepsilon_1\overset{\star}{,}h],\varepsilon_2)_a\\
=&-[h_a\overset{\star}{,}[\varepsilon_2\overset{\star}{,}\varepsilon_1]]-[\varepsilon_1\overset{\star}{,}[h_a\overset{\star}{,}\varepsilon_2]]-[\varepsilon_2\overset{\star}{,}[\varepsilon_1\overset{\star}{,}h_a]]\\
=&0
\end{aligned}
\tag{2.169}
$$

which vanishes by the virtue of the Moyal bracket satisfying the Jacobi identity.
At input degree -2 we have as inputs either $\varepsilon_1,\varepsilon_2,\mathcal{F}$ or ε_1,h,h. The first option gives

$$
\begin{aligned}
0 =&\ell_1(\ell_3(\varepsilon_1,\varepsilon_2,\mathcal{F}))\\
&+\ell_3(\ell_1(\varepsilon_1),\varepsilon_2,\mathcal{F})+\ell_3(\varepsilon_1,\ell_1(\varepsilon_2),\mathcal{F})+\ell_3(\varepsilon_1,\varepsilon_2,\ell_1(\mathcal{F}))\\
&+\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),\mathcal{F})+\ell_2(\ell_2(\varepsilon_2,\mathcal{F}),\varepsilon_1)+\ell_2(\ell_2(\mathcal{F},\varepsilon_1),\varepsilon_2)\,.
\end{aligned}
\tag{2.170}
$$

The first line vanishes due to (2.149), the third term in the second line vanishes due to (2.149), the third line vanishes again due to the Jacobi identity of the Moyal commutator and we are left with

$$0 = \ell_3(\ell_1(\varepsilon_1), \varepsilon_2, \mathcal{F}) + \ell_3(\varepsilon_1, \ell_1(\varepsilon_2), \mathcal{F}) \tag{2.171}$$

The second possibility at degree -2 is

$$\begin{aligned} 0 =&\ell_1(\ell_3(h_1, h_2, \varepsilon_3)) \\ &+\ell_3(\ell_1(h_1), h_2, \varepsilon_3) - \ell_3(h_1, \ell_1(h_2), \varepsilon_3) + \ell_3(h_1, h_2, \ell_1(\varepsilon_3)) \\ &+\ell_2(\ell_2(h_1, h_2), \varepsilon_3) - \ell_2(\ell_2(h_2, \varepsilon_3), h_1) + \ell_2(\ell_2(\varepsilon_3, h_1), h_2)\,. \end{aligned} \tag{2.172}$$

The first line vanishes due to (2.137), and we have

$$\begin{aligned} -\ell_3(h_1, h_2, \ell_1(\varepsilon_3)) =&\ell_3(\ell_1(h_1), h_2, \varepsilon_3) - \ell_3(h_1, \ell_1(h_2), \varepsilon_3) \\ &+\ell_2(\ell_2(h_1, h_2), \varepsilon_3) - \ell_2(\ell_2(h_2, \varepsilon_3), h_1) + \ell_2(\ell_2(\varepsilon_3, h_1), h_2)\,. \end{aligned} \tag{2.173}$$

On the left hand side we will have due to (2.142)

$$\begin{aligned} -\ell_3(h_1, h_2, \partial\varepsilon_3)_a =&[h_1^b \stackrel{\star}{,} [h_{2a} \stackrel{\star}{,} \partial_b\varepsilon_3]] + [h_2^b \stackrel{\star}{,} [\partial_a\varepsilon_3 \stackrel{\star}{,} h_{1b}]] + [\partial^b\varepsilon_3 \stackrel{\star}{,} [h_{1a} \stackrel{\star}{,} h_{2b}]] \\ &+ [h_2^b \stackrel{\star}{,} [h_{1a} \stackrel{\star}{,} \partial_b\varepsilon_3]] + [h_1^b \stackrel{\star}{,} [\partial_a\varepsilon_3 \stackrel{\star}{,} h_{2b}]] + [\partial^b\varepsilon_3 \stackrel{\star}{,} [h_{2a} \stackrel{\star}{,} h_{1b}]]\,, \end{aligned} \tag{2.174}$$

while the last three terms on the right hand side give

$$\begin{aligned} &\ell_2(\ell_2(h_1, h_2), \varepsilon_3)_a - \ell_2(\ell_2(h_2, \varepsilon_3), h_1)_a + \ell_2(\ell_2(\varepsilon_3, h_1), h_2)_a = \\ &=[\varepsilon_3 \stackrel{\star}{,} \partial^b[h_{1b} \stackrel{\star}{,} h_{2a}]] + [\varepsilon_3 \stackrel{\star}{,} \partial^b[h_{2b} \stackrel{\star}{,} h_{1a}]] + [\varepsilon_3 \stackrel{\star}{,} [h_1^b \stackrel{\star}{,} \partial_b h_{2a} - \partial_a h_{2b}]] + [\varepsilon_3 \stackrel{\star}{,} [h_2^b \stackrel{\star}{,} \partial_b h_{1a} - \partial_a h_{1b}]] \\ &- \Big(\partial^b[[\varepsilon_3 \stackrel{\star}{,} h_{2b}] \stackrel{\star}{,} h_{1a}] + \partial^b[h_{1b} \stackrel{\star}{,} [\varepsilon_3 \stackrel{\star}{,} h_{2a}]] + [[\varepsilon_3 \stackrel{\star}{,} h_2^b] \stackrel{\star}{,} \partial_b h_{1a} - \partial_a h_{1b}] + [h_1^b \stackrel{\star}{,} \partial_b[\varepsilon_3 \stackrel{\star}{,} h_{2a}] - \partial_a[\varepsilon_3 \stackrel{\star}{,} h_{2b}]]\Big) \\ &- \Big(\partial^b[[\varepsilon_3 \stackrel{\star}{,} h_{1b}] \stackrel{\star}{,} h_{2a}] + \partial^b[h_{2b} \stackrel{\star}{,} [\varepsilon_3 \stackrel{\star}{,} h_{1a}]] + [[\varepsilon_3 \stackrel{\star}{,} h_1^b] \stackrel{\star}{,} \partial_b h_{2a} - \partial_a h_{2b}] + [h_2^b \stackrel{\star}{,} \partial_b[\varepsilon_3 \stackrel{\star}{,} h_{1a}] - \partial_a[\varepsilon_3 \stackrel{\star}{,} h_{1b}]]\Big)\,. \end{aligned} \tag{2.175}$$

Even though the procedure is tedious, we can confirm that these two expressions are identical. Calculations can be facilitated by a computer [54]. We conclude

$$-\ell_3(h_1, h_2, \partial\varepsilon_3) = \ell_2(\ell_2(h_1, h_2), \varepsilon_3) - \ell_2(\ell_2(h_2, \varepsilon_3), h_1) + \ell_2(\ell_2(\varepsilon_3, h_1), h_2) \tag{2.176}$$

which leads us to the version of the $n = 3$ identity as

$$0 =\ell_3(\ell_1(h_1), h_2, \varepsilon_3) - \ell_3(h_1, \ell_1(h_2), \varepsilon_3)\,. \tag{2.177}$$

Both with (2.171), we see that it is consistent to set the products $\ell_3(\varepsilon, h, E) = 0$. Furthermore, we set all $\ell_4 = 0$.

n=4 With the products $\ell_4 = 0$, the $n = 4$ identity becomes

$$
\begin{aligned}
0 =& - \ell_2(\ell_3(x_1, x_2, x_3), x_4) + (-)^{\mathrm{x}_3\mathrm{x}_4}\ell_2(\ell_3(x_1, x_2, x_4), x_3) + (-)^{(1+\mathrm{x}_1)\mathrm{x}_2}\ell_2(x_2, \ell_3(x_1, x_3, x_4)) \\
&- (-)^{\mathrm{x}_1}\ell_2(x_1, \ell_3(x_2, x_3, x_4)) + \ell_3(\ell_2(x_1, x_2), x_3, x_4) + (-)^{1+\mathrm{x}_2\mathrm{x}_3}\ell_3(\ell_2(x_1, x_3), x_2, x_4) \\
&+ (-)^{\mathrm{x}_4(\mathrm{x}_2+\mathrm{x}_3)}\ell_3(\ell_2(x_1, x_4), x_2, x_3) - \ell_3(x_1, \ell_2(x_2, x_3), x_4) + (-)^{\mathrm{x}_3\mathrm{x}_4}\ell_3(x_1, \ell_2(x_2, x_4), x_3) \\
&+ \ell_3(x_1, x_2, \ell_2(x_3, x_4))
\end{aligned}
\tag{2.178}
$$

The degree of the identity is equal to $\deg(\ell_2\ell_3) = 1$, which means that a nontrivial result could be obtained for inputs of total degree -1 which is $\varepsilon\varepsilon\varepsilon h$, total degree -2 which are $\varepsilon\varepsilon\varepsilon F, \varepsilon\varepsilon hh$ or total degree -3 which are $\varepsilon\varepsilon hF, \varepsilon hhh$. Now, if we keep in mind that we have realized the product ℓ_3 as being non-vanishing only for the input of hhh and since $\deg \ell_2 = 0$, each option but the last (input of type εhhh) is trivial. The identity to check is

$$
\begin{aligned}
\ell_2(\ell_3(h_1, h_2, h_3), \varepsilon_4) =& \ell_3(\ell_2(h_1, \varepsilon_4), h_2, h_3) + \ell_3(h_1, \ell_2(h_2, \varepsilon_4), h_3) + \ell_3(h_1, h_2, \ell_2(h_3, \varepsilon_4)) \\
=& i\ell_3([\varepsilon_4 \overset{\star}{,} h_1], h_2, h_3) + i\ell_3(h_1, [\varepsilon_4 \overset{\star}{,} h_2], h_3) + i\ell_3(h_1, h_2, [\varepsilon_4 \overset{\star}{,} h_3]) \\
=& i\ell_3([\varepsilon_4 \overset{\star}{,} h_1], h_2, h_3) + i\ell_3([\varepsilon_4 \overset{\star}{,} h_2], h_1, h_3) + i\ell_3([\varepsilon_4 \overset{\star}{,} h_3], h_1, h_2)\,.
\end{aligned}
\tag{2.179}
$$

We simply find the left hand side as

$$
\begin{aligned}
i[\varepsilon_4 \overset{\star}{,} \ell_3(h_1, h_2, h_3)]_a =& - i[\varepsilon_4 \overset{\star}{,} [h_1^b \overset{\star}{,} [h_{2a} \overset{\star}{,} h_{3b}]]] - i[\varepsilon_4 \overset{\star}{,} [h_2^b \overset{\star}{,} [h_{3a} \overset{\star}{,} h_{1b}]]] - i[\varepsilon_4 \overset{\star}{,} [h_3^b \overset{\star}{,} [h_{1a} \overset{\star}{,} h_{2b}]]] \\
&- i[\varepsilon_4 \overset{\star}{,} [h_2^b \overset{\star}{,} [h_{1a} \overset{\star}{,} h_{3b}]]] - i[\varepsilon_4 \overset{\star}{,} [h_1^b \overset{\star}{,} [h_{3a} \overset{\star}{,} h_{2b}]]] - i[\varepsilon_4 \overset{\star}{,} [h_3^b \overset{\star}{,} [h_{2a} \overset{\star}{,} h_{1b}]]]\,,
\end{aligned}
\tag{2.180}
$$

while the right hand side gives us more work, but is straightforward

$$
\begin{aligned}
\ell_3([\varepsilon_4 \overset{\star}{,} h_1], h_2, h_3)_a =& - [[\varepsilon_4 \overset{\star}{,} h_1]^b \overset{\star}{,} [h_{2a} \overset{\star}{,} h_{3b}]] - [h_2^b \overset{\star}{,} [h_{3a} \overset{\star}{,} [\varepsilon_4 \overset{\star}{,} h_{1b}]]] - [h_3^b \overset{\star}{,} [[\varepsilon_4 \overset{\star}{,} h_{1a}] \overset{\star}{,} h_{2b}]] \\
&- [h_2^b \overset{\star}{,} [[\varepsilon_4 \overset{\star}{,} h_{1a}] \overset{\star}{,} h_{3b}]] - [[\varepsilon_4 \overset{\star}{,} h_1^b] \overset{\star}{,} [h_{3a} \overset{\star}{,} h_{2b}]] - [h_3^b \overset{\star}{,} [h_{2a} \overset{\star}{,} [\varepsilon_4 \overset{\star}{,} h_{1b}]]]
\end{aligned}
\tag{2.181}
$$

$$
\begin{aligned}
\ell_3([\varepsilon_4 \overset{\star}{,} h_2], h_1, h_3)_a =& - [[\varepsilon_4 \overset{\star}{,} h_2^b] \overset{\star}{,} [h_{1a} \overset{\star}{,} h_{3b}]] - [h_1^b \overset{\star}{,} [h_{3a} \overset{\star}{,} [\varepsilon_4 \overset{\star}{,} h_{2b}]]] - [h_3^b \overset{\star}{,} [[\varepsilon_4 \overset{\star}{,} h_{2a}] \overset{\star}{,} h_{1b}]] \\
&- [h_1^b \overset{\star}{,} [[\varepsilon_4 \overset{\star}{,} h_{2a}] \overset{\star}{,} h_{3b}]] - [[\varepsilon_4 \overset{\star}{,} h_2^b] \overset{\star}{,} [h_{3a} \overset{\star}{,} h_{1b}]] - [h_3^b \overset{\star}{,} [h_{1a} \overset{\star}{,} [\varepsilon_4 \overset{\star}{,} h_{2b}]]]
\end{aligned}
\tag{2.182}
$$

$$
\begin{aligned}
\ell_3([\varepsilon_4 \overset{\star}{,} h_3], h_1, h_2)_a =& - [[\varepsilon_4 \overset{\star}{,} h_3^b] \overset{\star}{,} [h_{1a} \overset{\star}{,} h_{2b}]] - [h_1^b \overset{\star}{,} [h_{2a} \overset{\star}{,} [\varepsilon_4 \overset{\star}{,} h_{3b}]]] - [h_2^b \overset{\star}{,} [[\varepsilon_4 \overset{\star}{,} h_{3a}] \overset{\star}{,} h_{1b}]] \\
&- [h_1^b \overset{\star}{,} [[\varepsilon_4 \overset{\star}{,} h_{3a}] \overset{\star}{,} h_{2b}]] - [[\varepsilon_4 \overset{\star}{,} h_3^b] \overset{\star}{,} [h_{2a} \overset{\star}{,} h_{1b}]] - [h_2^b \overset{\star}{,} [h_{1a} \overset{\star}{,} [\varepsilon_4 \overset{\star}{,} h_{3b}]]]\,.
\end{aligned}
\tag{2.183}
$$

Again, though the calculation is tedious, it is straightforward and can be assisted by a computer. The identity (2.179) is satisfied.

n=5 The identity with 5 inputs is schematically

$$\ell_1\ell_5 + \ell_2\ell_4 + \ell_3\ell_3 = 0\,. \tag{2.184}$$

We set all $\ell_{k\geq 4} = 0$, so the only thing to check in this identity is the combination of ℓ_3 products. However, the ℓ_3 is only non-vanishing for inputs of degree -1, in which case $\deg \ell_3(h,h,h) = -2$. This becomes an input to the second ℓ_3 and gives a zero by definition, so the identity is satisfied.

n≥6 The identities with 6 inputs will exclusively always have a product with more than 3 inputs, which are all 0 by definition.

We conclude that the MHSYM theory is an instance of the L_∞ algebra with the only non-vanishing products given by (2.128 - 2.134).

2.4.4 L_∞ structure of MHSYM theory: MHS covariant approach

In the covariant approach, we use the MHS vielbein as the fundamental field. The designation of elements to graded vector spaces is

- X_0 contains gauge parameters $\varepsilon(x,u)$, $\deg\varepsilon = 0$.
- X_{-1} contains fields $e_a(x,u)$, $\deg e_a = -1$.
- X_{-2} contains equations of motion $\mathcal{F}_a(x,u)$, $\deg\mathcal{F}_a = -2$.

The L_∞ structure is now simpler, with the only non-vanishing products being

$$\ell_2(\varepsilon_1,\varepsilon_2) = i[\varepsilon_2 \overset{\star}{,} \varepsilon_1] \tag{2.185}$$

$$\ell_2(\varepsilon,e)_a = i[e_a \overset{\star}{,} \varepsilon] \tag{2.186}$$

$$\ell_2(\mathcal{F},\varepsilon)_a = i[\varepsilon \overset{\star}{,} \mathcal{F}_a] \tag{2.187}$$

$$\begin{aligned}\ell_3(e_1,e_2,e_3)_a = &-[e_1^b \overset{\star}{,} [e_{2a} \overset{\star}{,} e_{3b}]] - [e_2^b \overset{\star}{,} [e_{3a} \overset{\star}{,} e_{1b}]] - [e_3^b \overset{\star}{,} [e_{1a} \overset{\star}{,} e_{2b}]] \\ &- [e_2^b \overset{\star}{,} [e_{1a} \overset{\star}{,} e_{3b}]] - [e_1^b \overset{\star}{,} [e_{3a} \overset{\star}{,} e_{2b}]] - [e_3^b \overset{\star}{,} [e_{2a} \overset{\star}{,} e_{1b}]]\end{aligned} \tag{2.188}$$

Proof: identifying products

The gauge transformations are

$$\delta_\varepsilon e_a = i[e_a, \varepsilon] \tag{2.189}$$

so we read out

$$\ell_1(\varepsilon)_a = 0, \quad \ell_2(\varepsilon, e)_a = i[e_a, \varepsilon], \quad \ell_{k>2}(\varepsilon, e, e, ...)_a = 0\,. \tag{2.190}$$

The commutator of gauge transformations is

$$[\delta_{\varepsilon_1}\delta_{\varepsilon_2} - \delta_{\varepsilon_2}\delta_{\varepsilon_1}]e_a = \delta_{i[\varepsilon_1 \overset{\star}{,} \varepsilon_2]} e_a \tag{2.191}$$

$$= \ell_2(\varepsilon_1, \ell_2(\varepsilon_2, e))_a - \ell_2(\varepsilon_2, \ell_2(\varepsilon_1, e))_a \tag{2.192}$$

$$= \delta_{\ell_2(\varepsilon_2, \varepsilon_1)} e_a \tag{2.193}$$

and we find

$$\ell_2(\varepsilon_1, \varepsilon_2) = i[\varepsilon_2 \overset{\star}{,} \varepsilon_1]\,. \tag{2.194}$$

Equations of motion are

$$-\,[e_b \overset{\star}{,} [e^b \overset{\star}{,} e_a]] = 0\,. \tag{2.195}$$

So we have

$$\ell_1(e)_a = 0, \quad \ell_2(e, e)_a = 0, \quad \ell_3(e, e, e)_a = 3![e^b \overset{\star}{,} [e_b \overset{\star}{,} e_a]]\,. \tag{2.196}$$

The non-diagonal version of ℓ_3 is

$$\ell_3(e_1, e_2, e_3)_a = [e_1^b \overset{\star}{,} [e_{2b} \overset{\star}{,} e_{3a}]] + [e_2^b \overset{\star}{,} [e_{3b} \overset{\star}{,} e_{1a}]] + [e_3^b \overset{\star}{,} [e_{1b} \overset{\star}{,} e_{2a}]] \tag{2.197}$$

$$+ [e_2^b \overset{\star}{,} [e_{1b} \overset{\star}{,} e_{3a}]] + [e_1^b \overset{\star}{,} [e_{3b} \overset{\star}{,} e_{2a}]] + [e_3^b \overset{\star}{,} [e_{2b} \overset{\star}{,} e_{1a}]]\,. \tag{2.198}$$

We set all $\ell_{k>3} = 0$

Proof: verifying identities

n=1

$$\ell_1^2 = 0 \tag{2.199}$$

which is identically true since all $\ell_1 = 0$

n=2

$$\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-)^{\mathrm{x}_1}\ell_2(x_1, \ell_1(x_2)) \tag{2.200}$$

The identity is trivially satisfied, again since $\ell_1 = 0$.

n=3

$$
\begin{aligned}
0 =&\ell_1(\ell_3(x_1,x_2,x_3))\\
&+\ell_3(\ell_1(x_1),x_2,x_3)+(-)^{\mathrm{x}_1}\ell_3(x_1,\ell_1(x_2),x_3)+(-)^{(\mathrm{x}_1+\mathrm{x}_2)}\ell_3(x_1,x_2,\ell_1(x_3))\\
&+\ell_2(\ell_2(x_1,x_2),x_3)+(-)^{\mathrm{x}_1(\mathrm{x}_2+\mathrm{x}_3)}\ell_2(\ell_2(x_2,x_3),x_1)+(-)^{(\mathrm{x}_2+\mathrm{x}_1)\mathrm{x}_3}\ell_2(\ell_2(x_3,x_1),x_2)
\end{aligned}
\tag{2.201}
$$

At input degree 0 we find the Jacobi identity of the Moyal bracket

$$
\begin{aligned}
0 =&\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),\varepsilon_3)+\ell_2(\ell_2(\varepsilon_2,\varepsilon_3),\varepsilon_1)+\ell_2(\ell_2(\varepsilon_3,\varepsilon_1),\varepsilon_2)\\
=&-[\varepsilon_3 \overset{\star}{,} [\varepsilon_2, \overset{\star}{,} \varepsilon_1]]-[\varepsilon_1 \overset{\star}{,} [\varepsilon_3, \overset{\star}{,} \varepsilon_2]]-[\varepsilon_2 \overset{\star}{,} [\varepsilon_1, \overset{\star}{,} \varepsilon_3]]\\
=&0\,.
\end{aligned}
\tag{2.202}
$$

At input degree -1 with inputs $\varepsilon_1\varepsilon_2 e_3$ we find again the Jacobi identity of the Moyal bracket

$$
\begin{aligned}
0 =&\,\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),e_3)_a+\ell_2(\ell_2(\varepsilon_2,e_3),\varepsilon_1)_a+\ell_2(\ell_2(e_3,\varepsilon_1),\varepsilon_2)_a\\
=&-[e_{3a} \overset{\star}{,} [\varepsilon_2, \overset{\star}{,} \varepsilon_1]]-[\varepsilon_1 \overset{\star}{,} [e_{3a}, \overset{\star}{,} \varepsilon_2]]-[\varepsilon_2 \overset{\star}{,} [\varepsilon_1, \overset{\star}{,} e_{3a}]]\\
=&0\,.
\end{aligned}
\tag{2.203}
$$

At input degree -2 with inputs ε_1, e_2, e_3

$$
\begin{aligned}
0 =&\ell_2(\ell_2(\varepsilon_1,e_2),e_3)+\ell_2(\ell_2(e_2,e_3),\varepsilon_1)-\ell_2(\ell_2(e_3,\varepsilon_1),e_2)\\
=&0
\end{aligned}
\tag{2.204}
$$

since all $\ell_2(e,e)$ with $e\in X_{-1}$ are 0.
At input degree -2 with inputs $\varepsilon_1,\varepsilon_2,\mathcal{F}_3$

$$
0 =\ell_2(\ell_2(\varepsilon_1,\varepsilon_2),\mathcal{F}_3)+\ell_2(\ell_2(\varepsilon_2,\mathcal{F}_3),\varepsilon_1)+\ell_2(\ell_2(\mathcal{F}_3,\varepsilon_1),\varepsilon_2)\,. \tag{2.205}
$$

This identity will be satisfied if we choose the product

$$
\ell_2(\varepsilon,\mathcal{F})_a \equiv i[\mathcal{F}_a, \overset{\star}{,} \varepsilon] \tag{2.206}
$$

n=4 With all $\ell_{k>3}=0$ we have

$$
\begin{aligned}
0 =&-\ell_2(\ell_3(x_1,x_2,x_3),x_4)+(-)^{\mathrm{x}_3\mathrm{x}_4}\ell_2(\ell_3(x_1,x_2,x_4),x_3)+(-)^{(1+\mathrm{x}_1)\mathrm{x}_2}\ell_2(x_2,\ell_3(x_1,x_3,x_4))\\
&-(-)^{\mathrm{x}_1}\ell_2(x_1,\ell_3(x_2,x_3,x_4))+\ell_3(\ell_2(x_1,x_2),x_3,x_4)+(-)^{1+\mathrm{x}_2\mathrm{x}_3}\ell_3(\ell_2(x_1,x_3),x_2,x_4)\\
&+(-)^{\mathrm{x}_4(\mathrm{x}_2+\mathrm{x}_3)}\ell_3(\ell_2(x_1,x_4),x_2,x_3)-\ell_3(x_1,\ell_2(x_2,x_3),x_4)+(-)^{\mathrm{x}_3\mathrm{x}_4}\ell_3(x_1,\ell_2(x_2,x_4),x_3)\\
&+\ell_3(x_1,x_2,\ell_2(x_3,x_4))
\end{aligned}
\tag{2.207}
$$

The degree of the identity is 1, so the input degrees can be $-1, -2, -3$. However, again, notice that while ℓ_2 does not change degree, the only possibility where ℓ_3 is non vanishing is at inputs e_1, e_2, e_3, which means there is a single non-trivial check with inputs $e_1, e_2, e_3, \varepsilon_4$.

$$\begin{aligned} 0 = & -\ell_2(\ell_3(e_1, e_2, e_3), \varepsilon_4) + \ell_2(\ell_3(e_1, e_2, \varepsilon_4), e_3) + \ell_2(e_2, \ell_3(e_1, e_3, \varepsilon_4)) \\ & + \ell_2(e_1, \ell_3(e_2, e_3, \varepsilon_4)) + \ell_3(\ell_2(e_1, e_2), e_3, \varepsilon_4) + \ell_3(\ell_2(e_1, e_3), e_2, \varepsilon_4) \\ & + \ell_3(\ell_2(e_1, \varepsilon_4), e_2, e_3) - \ell_3(e_1, \ell_2(e_2, e_3), \varepsilon_4) + \ell_3(e_1, \ell_2(e_2, \varepsilon_4), e_3) \\ & + \ell_3(e_1, e_2, \ell_2(e_3, \varepsilon_4)) \,. \end{aligned} \tag{2.208}$$

After removing the vanishing products, we are left with

$$\ell_2(\ell_3(e_1, e_2, e_3), \varepsilon_4) = \ell_3(\ell_2(e_1, \varepsilon_4), e_2, e_3) + \ell_3(e_1, \ell_2(e_2, \varepsilon_4), e_3) + \ell_3(e_1, e_2, \ell_2(e_3, \varepsilon_4)) \,. \tag{2.209}$$

The left hand side is

$$\begin{aligned} \ell_2(\ell_3(e_1, e_2, e_3), \varepsilon_4)_a = & i[\varepsilon_4 \stackrel{\star}{,} [e_1^b \stackrel{\star}{,} [e_{2b} \stackrel{\star}{,} e_{3a}]]] + i[\varepsilon_4 \stackrel{\star}{,} [e_2^b \stackrel{\star}{,} [e_{3b} \stackrel{\star}{,} e_{1a}]]] + i[\varepsilon_4 \stackrel{\star}{,} [e_3^b \stackrel{\star}{,} [e_{1b} \stackrel{\star}{,} e_{2a}]]] \\ & + i[\varepsilon_4 \stackrel{\star}{,} [e_2^b \stackrel{\star}{,} [e_{1b} \stackrel{\star}{,} e_{3a}]]] + i[\varepsilon_4 \stackrel{\star}{,} [e_1^b \stackrel{\star}{,} [e_{3b} \stackrel{\star}{,} e_{2a}]]] + i[\varepsilon_4 \stackrel{\star}{,} [e_3^b \stackrel{\star}{,} [e_{2b} \stackrel{\star}{,} e_{1a}]]] \,, \end{aligned} \tag{2.210}$$

while on the right hand side we have

$$\begin{aligned} \ell_3(\ell_2(e_1, \varepsilon_4), e_2, e_3)_a = & [i[\varepsilon_4 \stackrel{\star}{,} e_1^b] \stackrel{\star}{,} [e_{2b} \stackrel{\star}{,} e_{3a}]] + [e_2^b \stackrel{\star}{,} [e_{3b} \stackrel{\star}{,} [\varepsilon_4 \stackrel{\star}{,} e_{1a}]]] + [e_3^b \stackrel{\star}{,} [[\varepsilon_4 \stackrel{\star}{,} e_{1b}] \stackrel{\star}{,} e_{2a}]] \\ & + [e_2^b \stackrel{\star}{,} [[\varepsilon_4 \stackrel{\star}{,} e_{1b}] \stackrel{\star}{,} e_{3a}]] + [[\varepsilon_4 \stackrel{\star}{,} e_1^b] \stackrel{\star}{,} [e_{3b} \stackrel{\star}{,} e_{2a}]] + [e_3^b \stackrel{\star}{,} [e_{2b} \stackrel{\star}{,} [\varepsilon_4 \stackrel{\star}{,} e_{1a}]]] \end{aligned} \tag{2.211}$$

$$\begin{aligned} \ell_3(\ell_2(e_2, \varepsilon_4), e_1, e_3)_a = & [i[\varepsilon_4 \stackrel{\star}{,} e_2^b] \stackrel{\star}{,} [e_{1b} \stackrel{\star}{,} e_{3a}]] + [e_1^b \stackrel{\star}{,} [e_{3b} \stackrel{\star}{,} [\varepsilon_4 \stackrel{\star}{,} e_{2a}]]] + [e_3^b \stackrel{\star}{,} [[\varepsilon_4 \stackrel{\star}{,} e_{2b}] \stackrel{\star}{,} e_{1a}]] \\ & + [e_1^b \stackrel{\star}{,} [[\varepsilon_4 \stackrel{\star}{,} e_{2b}] \stackrel{\star}{,} e_{3a}]] + [[\varepsilon_4 \stackrel{\star}{,} e_2^b] \stackrel{\star}{,} [e_{3b} \stackrel{\star}{,} e_{1a}]] + [e_3^b \stackrel{\star}{,} [e_{1b} \stackrel{\star}{,} [\varepsilon_4 \stackrel{\star}{,} e_{2a}]]] \end{aligned} \tag{2.212}$$

$$\begin{aligned} \ell_3(\ell_2(e_3, \varepsilon_4), e_1, e_2)_a = & [i[\varepsilon_4 \stackrel{\star}{,} e_3]^b \stackrel{\star}{,} [e_{1b} \stackrel{\star}{,} e_{2a}]] + [e_1^b \stackrel{\star}{,} [e_{2b} \stackrel{\star}{,} [\varepsilon_4 \stackrel{\star}{,} e_{3a}]]] + [e_2^b \stackrel{\star}{,} [[\varepsilon_4 \stackrel{\star}{,} e_{3b}] \stackrel{\star}{,} e_{1a}]] \\ & + [e_1^b \stackrel{\star}{,} [[\varepsilon_4 \stackrel{\star}{,} e_{3b}] \stackrel{\star}{,} e_{2a}]] + [[\varepsilon_4 \stackrel{\star}{,} e_3^b] \stackrel{\star}{,} [e_{2b} \stackrel{\star}{,} e_{1a}]] + [e_2^b \stackrel{\star}{,} [e_{1b} \stackrel{\star}{,} [\varepsilon_4 \stackrel{\star}{,} e_{3a}]]] \,. \end{aligned} \tag{2.213}$$

With a bit of help from a computer, facilitated by the fact that the Moyal product is associative, we can see that this identity is also satisfied.

n=5 Again, we have a similar conclusion that only the terms of the type $\ell_3\ell_3$ could be non-trivial. However, ℓ_3 are non-vanishing only for inputs e_1, e_2, e_3, but since $\ell_3(e_1, e_2, e_2) \in X_{-2}$, and any ℓ_3 with an input from X_{-2} is vanishing, the identity holds trivially.

n≥6 All identities will always contain terms which all have at least an ℓ_4 which is 0. These identities are thus trivially satisfied.

We confirm the conclusions of the calculation above, the MHSYM theory is described by an L_∞ structure. The covariant formulation allows for a simpler presentation of the L_∞ products, thus also the identities are easier to check. Though the formulations are equivalent in case of a Minkowski background $e_a = u_a + h_a$, the covariant formulation is more general and allows for different backgrounds. This is an important point to bear in mind for future work. As we have seen, the different phases of the theory are gauge inequivalent, and it would be expected that they are described by a different L_∞ structure. Here we found the L_∞ structure of the most general formulation, and it might prove interesting to find morphisms between L_∞ algebras of the various formulations.

Chapter 3

A unitary representation of the Lorentz group on Hermite functions

For the purposes of the expansion in chapter 4, we would like to represent the Lorentz group on the space of square integrable functions in 4 dimensions $L^2(\mathbb{R}^4)$ (generalization to an arbitrary number of dimensions is straightforward). We start with a reminder on representing groups on spaces of functions and outline the method we used to construct the representation. Afterwards, we specialize to 4 dimensions with Hermite functions forming the basis of the representation space. We provide explicit representation matrices for finite boosts in particular directions and rotations around a particular axis. From the finite case, we find the generators of the Lorentz Lie algebra. Finally, we discuss a basis for the vector space of Hermite functions diagonal in the rotation operator around the z axis.

The defining representation of the Lorentz group are matrices $\Lambda^{\mu}{}_{\nu}$ used to perform a continuous linear transformation on coordinate components

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{3.1}$$

such that the sum $\eta_{\mu\nu} x^{\mu} x^{\nu}$ is preserved, with $\eta_{\mu\nu}$ the metric of Minkowski spacetime

$$\Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta}. \tag{3.2}$$

The matrices Λ are finite dimensional (their dimension being determined by the dimension of the Minkowski spacetime), but they are not unitary, instead satisfying the condition (3.2) above.

We choose multi-dimensional Hermite functions to be the basis of the representation space. Since the group of rotations is a subgroup of the Lorentz group, the proposed method can be also used to represent $SO(3)$ on the space of $L^2(\mathbb{R}^3)$, or generalized to representing $SO(d)$ on $L^2(\mathbb{R}^d)$ spanned by Hermite functions. By choosing an inner product, the representation space can be made into a Hilbert space, and by construction the representation matrices will be unitary.

3.1 Representing a group on a function space

Let a group G be represented on $\mathbb{R}^d$ by linear operators $\rho(g)$ as

$$x \to x' = \rho(g) \cdot x \tag{3.3}$$

with the central dot representing matrix multiplication. When acting on the coordinate components x, we will omit writing $\rho(g)$ and simply write g. A group is represented on the space of functions on $\mathbb{R}^d$ as

$$h(x) \to h'(x) \equiv h(g^{-1}x) \tag{3.4}$$

since for $g_3 = g_2 \circ g_1$ we have

$$h''(x) = h'(g_2^{-1}x) = h(g_1^{-1}g_2^{-1}x) = h(g_3^{-1}x)\,. \tag{3.5}$$

Let us choose a complete orthonormal real basis of functions $f_N(x)$ which will span $L^2(\mathbb{R}^d)$. The capital letter indices can be multi-valued, making this exposition general. We will spell them out for the particular choice of Hermite functions in the next section. The orthonormality and completeness conditions are given by

$$\int d^dx\, f_N(x) f_M(x) = \delta_{NM}, \qquad \sum_N f_N(x) f_N(y) = \delta^{(d)}(x-y)\,. \tag{3.6}$$

We can now expand $h(x)$, an arbitrary element of $L^2(\mathbb{R}^d)$ in the chosen basis as

$$h(x) = \sum_N h^N f_N(x)\,. \tag{3.7}$$

This type of an expansion can now be used to relate the components h^N of the original function to the components of h'^N of a transformed function, since there are two ways of expanding a transformed function $h'(x) = h(g^{-1}x)$.

$$h'(x) = \sum_N h^N f_N(g^{-1}x) = \sum_M h'^M f_M(x) \tag{3.8}$$

We can multiply (3.8) with $f_K(x)$, and integrate over the whole space.

$$\sum_N h^N \int d^d x\, f_K(x) f_N(g^{-1}x) = \sum_M h'^M \int d^d x\, f_K(x) f_M(x) \tag{3.9}$$

Due to the orthogonality of the basis functions (3.6), we can conclude that the new components h'^M are linearly related to the old components h^N as

$$h'^M = \sum_N h^N \left(\int dx\, f_M(x) f_N(g^{-1}x) \right) \tag{3.10}$$

and thus define a representation of G denoted by

$$D^M_N(g) \equiv \int dx\, f_M(x) f_N(g^{-1}x)\,. \tag{3.11}$$

The transformation between function components can now be written as

$$h'^M = \sum_N D^M_N(g) h^N \tag{3.12}$$

The constructed matrices $D^M_N(g)$ are real since the choice of the basis functions $f_N(x)$ was real. The procedure can be generalized to complex valued functions.

It is easy to see that the matrices D^M_N form a representation of the group G, since for $g_3 = g_2 \circ g_1$ we have

$$\begin{aligned} h''(x) =& h'(g_2^{-1}x) = h(g_1^{-1}g_2^{-1}x) = h(g_3^{-1}x) && (3.13)\\ =& \sum_N h''^N f_N(x) && (3.14)\\ =& \sum_M h'^M f_M(g_2^{-1}x) = \sum_{M,K} D^M_K(g_1) h^K f_M(g_2^{-1}x) && (3.15)\\ =& \sum_M D^N_K(g_3) h^K f_N(x)\,. && (3.16) \end{aligned}$$

Again, due to the orthogonality of the basis functions, we can conclude

$$h''^N = \sum_{M,K} D^M_K(g_1) \left(\int dx\, f_N(x) f_M(g_2^{-1}x) \right) h^K = \sum_{M,K} D^M_K(g_1) D^N_M(g_2) h^K \tag{3.17}$$

meaning that the composition is respected

$$D^N_K(g_3) = \sum_M D^N_M(g_2) D^M_K(g_1)\,. \tag{3.18}$$

To prove unitarity, we can use equation (3.11) and see that the representation matrices enable us to relate transformed and non-transformed basis functions as

$$f_N(g^{-1}x) = \sum_M D^M_N(g) f_M(x)\,. \tag{3.19}$$

This expression makes it easy to prove that for the rotation group and the Lorentz group, the constructed representation is unitary. The following two integrals are equal in value

$$\int d^d x \, f_N(g^{-1}x) f_M(g^{-1}x) = \int d^d y \, f_N(y) f_M(y) = \delta_{NM} \tag{3.20}$$

because under the change of variables $y \equiv g^{-1}x$ the Jacobian is equal to unity. Applying the transformation (3.19) we get

$$\int dx \, f_N(g^{-1}x) f_M(g^{-1}x) = \sum_{J,K} D^J_N D^K_M \underbrace{\int dx \, f_J(x) f_K(x)}_{\delta_{JK}} = \sum_J D^J_N D^J_M \tag{3.21}$$

Meaning that the (real) representation matrices $D^M_N(g)$ are unitary.

$$\sum_J D^J_N D^J_M = \sum_J (D^T)^N_J D^J_M = \delta_{NM} \, . \tag{3.22}$$

It is important to note that one of the first constructions of a unitary infinite-dimensional representations of the Lorentz group was done by Dirac in 1944. [55]. The representation space was a suitably defined space of infinite sums of polynomials of a real variable with the coefficients in sums named "expansors".

3.2 Representation of the Lorentz group on $L^2(\mathbb{R}^4)$

3.2.1 Hermite functions and the generating integral

A good basis for the Hilbert space $L^2(\mathbb{R}^4)$ are multi-dimensional Hermite functions defined below. Partial results for the representation matrices of the Lorentz group on Hermite functions were obtained by Ruiz [56] and generalized by Rotbart [57] in the context of finding boosted eigenfunctions of a relativistic quantum harmonic oscillator. Their results cover the case of one-dimensional boosts. The main idea we take from their calculations is to focus on generating functions and find the sought-for result in the subsequent expansion. Our method is more general and enables us to calculate the representation matrices for arbitrary elements of the Lorentz group.

$H_n(x)$ are the (physicists') Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \, , \tag{3.23}$$

where the index n can attain arbitrary non-negative integer values, and from the polynomials, the Hermite functions are defined with a suitable normalization factor as

$$f_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_n(x) \,. \tag{3.24}$$

Importantly, they are orthonormal on the whole real line

$$\int_{-\infty}^{\infty} dx \, f_n(x) f_m(x) = \delta_{nm} \,. \tag{3.25}$$

The completeness identity for Hermite functions is given as

$$\sum_{n=0}^{\infty} f_n(x) f_n(y) = \delta(x-y) \,. \tag{3.26}$$

We define a multi-dimensional Hermite function as a product

$$f_{n_0 n_1 n_2 n_3}(t, x, y, z) \equiv f_{n_0}(t) f_{n_1}(x) f_{n_2}(y) f_{n_3}(z) \equiv f_{\mathbf{N}}(x) \,, \tag{3.27}$$

where we have defined a multi-index notation, $\mathbf{N} = \{n_0 n_1 n_2 n_3\}$. To calculate the representation matrices, as seen above in (3.11), we need to integrate:

$$D^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}(\Lambda) = \int d^4x \, f_{m_0 m_1 m_2 m_3}(x) f_{n_0 n_1 n_2 n_3}(\Lambda^{-1} x) \,. \tag{3.28}$$

A direct calculation of the above integral is a hard problem, but we can use an alternative route and find the result in the following way. The generating function for Hermite polynomials is given by

$$e^{2xq_1 - q_1^2} = \sum_{m_1=0}^{\infty} H_{m_1}(x) \frac{q_1^{m_1}}{m_1!} \,. \tag{3.29}$$

Multiplying it by $e^{-x^2/2}$, we define the generating function for Hermite functions

$$E_1(x, q_1) \equiv e^{2xq_1 - q_1^2 - x^2/2} = \sum_{m_1=0}^{\infty} c_{m_1} \frac{q_1^{m_1}}{m_1!} f_{m_1}(x) \tag{3.30}$$

where $c_{m_1} = (2^{m_1} m_1! \sqrt{\pi})^{1/2}$. We can easily generalize the new generating function to 4 dimensions

$$E(x, q) \equiv E_0(t, q_0) E_1(x, q_1) E_2(y, q_2) E_3(z, q_3) \,. \tag{3.31}$$

Next, we multiply two generating functions (3.31), integrate the product, and expand it in the generating variables in the following way:

$$\int d^4x \, E(x, q) E(\Lambda^{-1} x, p) = \sum_{\mathbf{M}=0}^{\infty} \sum_{\mathbf{N}=0}^{\infty} c_{\mathbf{M}} c_{\mathbf{N}} \frac{q^{\mathbf{M}}}{\mathbf{M}!} \frac{p^{\mathbf{N}}}{\mathbf{N}!} \int d^4x \, f_{\mathbf{M}}(x) f_{\mathbf{N}}(\Lambda^{-1} x) \tag{3.32}$$

where $\mathbf{N}, \mathbf{M}$ are multi indices standing for $\mathbf{N} = \{n_0, n_1, n_2, n_3\}$, $\mathbf{M} = \{m_0, m_1, m_2, m_3\}$. The coefficients on the right hand side of (3.32) provide exactly the transformation matrices (3.28). It will be shown below that the integral on the left hand side of (3.32) is relatively easily obtainable for any Lorentz transformation. In [56, 57] they have employed light-cone coordinates and evaluated this integral for the case of one-dimensional boosts.

3.2.2 Generating integral for an arbitrary Lorentz transformation

The idea is to rewrite the integral (3.32) in the general 4-dimensional case for any Lorentz transformation in the form of a Gaussian integral. To proceed in a compact fashion, we introduce auxiliary variables

$$p^\mu \equiv (-p_0, p_1, p_2, p_3), \quad q^\mu \equiv (-q_0, q_1, q_2, q_3), \quad n^\mu \equiv (1, 0, 0, 0)\,. \tag{3.33}$$

Even though they are written in a 4-vector form, by our demand, they do not change under Lorentz transformations. Their sole purpose is to be a placeholder, and enable 4-vector notation[1], through which we can rewrite the integral (3.32) in a matrix form.

For the 4-dimensional case, we write the generating function (3.31) for the Hermite functions as

$$E(x, q) = \exp\left[2x_\mu q^\mu - q_\mu q^\mu - 2\left(n_\mu q^\mu\right)^2 - \frac{1}{2}\left(x_\mu x^\mu + 2\left(n_\mu x^\mu\right)^2\right)\right]. \tag{3.34}$$

We repeat for some transformed variables $x' = \Lambda^{-1} x$

$$E(x', p) = \exp\left[2x'_\mu p^\mu - p_\mu p^\mu - 2\left(n_\mu p^\mu\right)^2 - \frac{1}{2}\left(x'_\mu x'^\mu + 2\left(n_\mu x'^\mu\right)^2\right)\right]. \tag{3.35}$$

After a careful rearranging of the terms, the integral (3.32) can be rewritten as

$$\int d^4x\, E(x, q) E(x', p) = \int d^4x \, \exp[-\frac{1}{2} x^\alpha A_{\alpha\beta} x^\beta + J_\alpha x^\alpha + C] \tag{3.36}$$

with

$$\begin{aligned} A_{\alpha\beta} &= 2\eta_{\alpha\beta} + 2(\Lambda^{-1})_{0\alpha}(\Lambda^{-1})_{0\beta} + 2n_\alpha n_\beta && (3.37)\\ J_\alpha &= 2p^\nu (\Lambda^{-1})_{\nu\alpha} + 2q^\nu \eta_{\nu\alpha} && (3.38)\\ C &= -\, q_\mu q^\mu - 2\left(n_\mu q^\mu\right)^2 - p_\mu p^\mu - 2\left(n_\mu p^\mu\right)^2 \,. && (3.39) \end{aligned}$$

[1] eg. $t^2 + x^2 + y^2 + z^2 = x_\mu x^\mu + 2(n_\mu x^\mu)^2$

The result of the integral can now easily be obtained since it is of the Gaussian form (see e.g. ch. 9 in [58])

$$
\begin{aligned}
I(p,q,\Lambda^{-1}) &= \int d^4x\, E_1(x,q)E_2(x',p) && (3.40)\\
&= \frac{(2\pi)^2}{\sqrt{\det A}} \exp\left[\frac{1}{2} J_\alpha [A^{-1}]^{\alpha\beta} J_\beta\right] e^{-(q_0^2+\ldots+q_3^2+p_0^2+\ldots+p_3^2)}\,. && (3.41)
\end{aligned}
$$

We notice that the matrix $A_{\alpha\beta}$ contains only the 0th rows of the Lorentz transformation matrix, where we find only information about boosts, i.e. the matrix $A_{\alpha\beta}$ is insensitive to the rotation part of the Lorentz group. For that reason, only inside $A_{\alpha\beta}$, we can use an explicit expression for the inverse of Lorentz boosts

$$
(\Lambda^{-1})^\alpha{}_\beta = \begin{bmatrix}
\gamma & -\gamma v_x/c & -\gamma v_y/c & -\gamma v_z/c \\
-\gamma v_x/c & 1+(\gamma-1)\frac{v_x^2}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} \\
-\gamma v_y/c & (\gamma-1)\frac{v_y v_x}{v^2} & 1+(\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\
-\gamma v_z/c & (\gamma-1)\frac{v_z v_x}{v^2} & (\gamma-1)\frac{v_z v_y}{v^2} & 1+(\gamma-1)\frac{v_z^2}{v^2}
\end{bmatrix} \tag{3.42}
$$

This enables us to explicitly write down $A_{\alpha\beta}$.

$$
A = 2\gamma^2 \begin{pmatrix}
1 & -v_x & -v_y & -v_z \\
-v_x & v_x^2+\frac{1}{\gamma^2} & v_x v_y & v_x v_z \\
-v_y & v_x v_y & v_y^2+\frac{1}{\gamma^2} & v_y v_z \\
-v_z & v_x v_z & v_y v_z & v_z^2+\frac{1}{\gamma^2}
\end{pmatrix} \tag{3.43}
$$

The determinant of $A_{\alpha\beta}$ is given by $\det A = 16\,\gamma^2$, while the inverse is

$$
A^{-1} = \begin{pmatrix}
\frac{1}{2} & \frac{v_x}{2} & \frac{v_y}{2} & \frac{v_z}{2} \\
\frac{v_x}{2} & \frac{1}{2} & 0 & 0 \\
\frac{v_y}{2} & 0 & \frac{1}{2} & 0 \\
\frac{v_z}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix} \tag{3.44}
$$

We can also explicitly write down the inverse of $A_{\alpha\beta}$ as

$$
[A^{-1}]^{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta} + n^\alpha n^\beta + \frac{1}{4\gamma^2}\left((\Lambda^{-1})^{0\alpha}(\Lambda^{-1})^{0\beta} - (\Lambda^{-1})^{\alpha 0}(\Lambda^{-1})^{\beta 0}\right) \tag{3.45}
$$

To shorten the formulas, define

$$
Z^{\alpha\beta} = \frac{1}{4\gamma^2}\left((\Lambda^{-1})^{0\alpha}(\Lambda^{-1})^{0\beta} - (\Lambda^{-1})^{\alpha 0}(\Lambda^{-1})^{\beta 0}\right) \tag{3.46}
$$

The generating integral (3.40) is now given by

$$
\begin{aligned}
I(p,q,\Lambda^{-1}) = \frac{\pi^2}{\gamma} & \exp\left[2p^\mu p^\nu\left((\Lambda^{-1})_{\nu 0}(\Lambda^{-1})_{\mu 0} + Z^{\alpha\beta}(\Lambda^{-1})_{\mu\beta}(\Lambda^{-1})_{\nu\alpha} - n_\mu n_\nu\right)\right] \\
& \times \exp\left[2q^\mu q^\nu\, Z_{\mu\nu}\right] \\
& \times \exp\left[2p^\mu q^\nu\left((\Lambda^{-1})_{\mu\nu} + 2(\Lambda^{-1})_{\mu 0}n_\nu + Z_{\alpha\nu}(\Lambda^{-1})_\mu{}^\alpha + Z_{\nu\alpha}(\Lambda^{-1})_\mu{}^\alpha\right)\right]
\end{aligned}
\tag{3.47}
$$

The formula (3.47) is the first important result of this chapter. It is possible to extract from this integral a transformation matrix for an arbitrary Lorentz transformation. For the intelligibility of expressions, we will provide formulas for boosts and rotations separately.

A nice consistency check is to calculate $I(p,q,\Lambda^{-1})$ for the case of one-dimensional boosts, eg. $v^x = v \neq 0$ while $v^y = v^z = 0$. That case was covered by [57] in their formula (10). To get a two-dimensional result we can set.

$$
p^\mu = (-p_0, p_1, 0, 0), \quad q^\mu = (-q_0, q_1, 0, 0)\,. \tag{3.48}
$$

The integral

$$
I(p,q,v) = \pi^2\sqrt{1-v^2}\exp\left[-2p_0p_1v + 2q_0q_1v + 2p_0q_0\sqrt{1-v^2} + 2p_1q_1\sqrt{1-v^2}\right] \tag{3.49}
$$

agrees completely with [57] after appropriate identifications.

3.2.3 Extracting the representation matrices

From the expression (3.32) we can see that to extract the representation matrices, we will have to expand $I(q,p,\Lambda^{-1})$ in powers of p and q.

$$
I(p,q,\Lambda^{-1}) = \sum_{\mathbf{N},\mathbf{M}} \frac{p^{\mathbf{N}}}{\mathbf{N}!}\frac{q^{\mathbf{M}}}{\mathbf{M}!}\left[\frac{\partial^{\mathbf{N}+\mathbf{M}} I(p,q,\Lambda^{-1})}{\partial p^{\mathbf{N}}\partial q^{\mathbf{M}}}\right]\Bigg|_{p=q=0} \tag{3.50}
$$

The multi index notation was defined above[2]. The representation matrices are then given by

$$D_{\mathbf{N}}^{\mathbf{M}}(\Lambda) = c_{\mathbf{M}} c_{\mathbf{N}} \frac{\partial^{\mathbf{N}+\mathbf{M}} I(p,q,\Lambda^{-1})}{\partial p^{\mathbf{N}} \partial q^{\mathbf{M}}}\Bigg|_{p=q=0} \tag{3.53}$$

where the $c_{\mathbf{M}}$ are multi-dimensional normalization constants defined through generalizing (3.24).

An alternative calculation to performing the partial derivatives starts by noticing that

$$I(p,q,\Lambda^{-1}) = \frac{\pi^2}{\gamma} e^{f(p,q,\Lambda^{-1})} = \frac{\pi^2}{\gamma} \sum_r \frac{1}{r!} f(p,q,\Lambda^{-1})^r \,. \tag{3.54}$$

To get $D_{\mathbf{N}}^{\mathbf{M}}(\Lambda)$ we can write down an expression for $f(p,q,\lambda)^r$, identify the term containing $p^{\mathbf{N}} q^{\mathbf{M}}$ and multiply with appropriate constants. There is always a unique r for a certain combination of $\mathbf{M}, \mathbf{N}$.

3.2.4 Representation matrices for boosts

For a boost in a single direction (e.g. x) with velocity v, we find the same result as in [57] but generalized to 4 dimensions

$$I(p,q,v) = \frac{\pi^2}{\gamma} \exp\left[2(q_0 q_1 - p_0 p_1)v + 2(p_0 q_0 + p_1 q_1)\sqrt{1-v^2} + 2p_2 q_2 + 2p_3 q_3\right] \,. \tag{3.55}$$

Following the procedure explained in the previous paragraph, we can identify the coefficients in the expansion over p and q. The transformation matrices for boosts in the x direction are given by

$$\begin{aligned} D_{n_0 n_1 n_2 n_3}^{m_0 m_1 m_3 m_3}(v\hat{x}) = {} & \delta_{-n_0+n_1+n_2+n_3}^{-m_0+m_1+m_2+m_3} \delta_{n_2}^{m_2} \delta_{n_3}^{m_3} \sqrt{\frac{m_1! n_0!}{n_1! m_0!}} \times \\ & \sum_{j=0}^{m_0} \binom{m_0}{j} \binom{n_1}{m_1 - j} (-1)^{n_1 - m_1 + j} \sqrt{1-v^2}^{m_1+m_0+1-2j} v^{2j-m_1+n_1} \,. \end{aligned} \tag{3.56}$$

[2]To be precise, we report the formula (3.50) with all the indices spelled out

$$I(p,q,\Lambda^{-1}) = \sum_{\{n\},\{m\}=0}^{\infty} \frac{p_0^{n_0}}{n_0!} \frac{p_1^{n_1}}{n_1!} \frac{p_2^{n_2}}{n_2!} \frac{p_3^{n_3}}{n_3!} \frac{q_0^{m_0}}{m_0!} \frac{q_1^{m_1}}{m_1!} \frac{q_2^{m_2}}{m_2!} \frac{q_3^{m_3}}{m_3!} \times \tag{3.51}$$

$$\left[\frac{\partial^{n_0+n_1+n_2+n_3+m_0+m_1+m_2+m_3}}{\partial p_0^{n_0} \partial p_1^{n_1} \partial p_2^{n_2} \partial p_3^{n_3} \partial q_0^{m_0} \partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} I(p,q,\Lambda^{-1})\right]\Bigg|_{p=q=0} \tag{3.52}$$

We can easily also find particular cases for boosts in y and z directions

$$D^{m_0 m_1 m_3 m_3}_{n_0 n_1 n_2 n_3}(v\hat{y}) = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_1}_{n_1}\delta^{m_3}_{n_3}\sqrt{\frac{m_2!n_0!}{n_2!m_0!}}\times$$
$$\sum_{j=0}^{m_0}\binom{m_0}{j}\binom{n_2}{m_2-j}(-1)^{n_2-m_2+j}\sqrt{1-v^2}^{m_2+m_0+1-2j}v^{2j-m_2+n_2} \quad (3.57)$$

$$D^{m_0 m_1 m_3 m_3}_{n_0 n_1 n_2 n_3}(v\hat{z}) = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\sqrt{\frac{m_3!n_0!}{n_3!m_0!}}\times$$
$$\sum_{j=0}^{m_0}\binom{m_0}{j}\binom{n_3}{m_3-j}(-1)^{n_3-m_3+j}\sqrt{1-v^2}^{m_3+m_0+1-2j}v^{2j-m_3+n_3}. \quad (3.58)$$

The matrices (3.56-3.57) are unitary by construction, and they are infinite dimensional as each index n, m ranges from 0 to ∞. The number $N = -n_0 + n_1 + n_2 + n_3$ is invariant, and can be used to reduce the matrices into sectors labeled by N. It is however easy to see that each such sector is in itself infinite dimensional; there is an infinite number of ways to combine one negative and three positive integers and obtain the same N. For example, consider a function $f_{0000}(x)$, and boost it in the x direction. We can apply the transformation rule (3.19)

$$f'_{0000}(x) = f_{0000}(\Lambda^{-1}x) = \sum_{m_0=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty} D^{m_0 m_1 m_3 m_3}_{0\,0\,0\,0}(v) f_{m_0 m_1 m_2 m_3}(x) \quad (3.59)$$
$$=\sqrt{1-v^2}\sum_{m_0=0}^{\infty}\sum_{m_1=0}^{\infty}\delta^{m_1}_{m_0}v^{m_1}f_{m_0 m_1 00} \quad (3.60)$$
$$=\sqrt{1-v^2}\left(f_{0000} + v f_{1100} + v^2 f_{2200} + \dots\right) \quad (3.61)$$

and find a truly infinite sum on the right hand side. As a curiosity we note that this particular example has an interesting relation to the Mehler's formula [59]. We rewrite

$$f_{0000}(\Lambda^{-1}x) = f_0(y)f_0(z)\sqrt{1-v^2}\sum_{n=0}^{\infty}v^n f_n(x)f_n(t) \quad (3.62)$$

and recognize on the right hand side

$$\sum_{n=0}^{\infty}v^n f_n(x)f_n(t) = \frac{1}{\sqrt{\pi}\sqrt{1-v^2}}\exp\left(-\frac{1-v}{1+v}\frac{(x+t)^2}{4} - \frac{1+v}{1-v}\frac{(x-t)^2}{4}\right). \quad (3.63)$$

We emphasize again that our method defines a homomorphism from any Lorentz transformation matrix Λ to the representation $D^{\mathbf{M}}_{\mathbf{N}}(\Lambda)$. In the general case of a boost parametrized by $\vec{v} = (v_x, v_y, v_z)$, the integral (3.32) gives

$$
\begin{aligned}
I = \pi^2 \sqrt{1-\vec{v}^2} \exp \Bigg[2 \Big(& p_0 q_0 \sqrt{1-\vec{v}^2} - p_0 p_1 v_x - p_0 p_2 v_y - p_0 p_3 v_z + q_0 q_1 v_x + q_0 q_2 v_y + q_0 q_3 v_z \\
& + p_1 q_1 \frac{v_y^2 + v_z^2 + v_x^2 \sqrt{1-\vec{v}^2}}{\vec{v}^2} - p_1 q_2 \frac{v_x v_y}{\vec{v}^2}(1-\sqrt{1-\vec{v}^2}) - p_1 q_3 \frac{v_x v_z}{\vec{v}^2}(1-\sqrt{1-\vec{v}^2}) \\
& - p_2 q_1 \frac{v_x v_y}{\vec{v}^2}(1-\sqrt{1-\vec{v}^2}) + p_2 q_2 \frac{v_x^2 + v_z^2 + v_y^2 \sqrt{1-\vec{v}^2}}{\vec{v}^2} - p_2 q_3 \frac{v_y v_z}{\vec{v}^2}(1-\sqrt{1-\vec{v}^2}) \\
& - p_3 q_1 \frac{v_x v_z}{\vec{v}^2}(1-\sqrt{1-\vec{v}^2}) - p_3 q_2 \frac{v_y v_z}{\vec{v}^2}(1-\sqrt{1-\vec{v}^2}) + p_3 q_3 \frac{v_x^2 + v_y^2 + v_z^2 \sqrt{1-\vec{v}^2}}{\vec{v}^2} \Big) \Bigg] .
\end{aligned}
\tag{3.64}
$$

from which it is possible to extract the representation matrix $D^{\mathbf{M}}_{\mathbf{N}}(\Lambda)$ through the same procedure as above. The explicit expressions in the general case are somewhat more complicated.

3.2.5 Representation matrices for rotations

In the case of pure rotations without boosts, we use $(\Lambda^{-1})_{\mu\nu} = (R)_{\mu\nu}$, with $R_{00} = -1, R_{0i} = R_{i0} = 0, R_{ij} \neq 0$, where R_{ij} is an orthogonal rotation matrix. The integral (3.32) attains a very simple form

$$
I = \pi^2 \exp\left[2p_0 q_0\right] \exp\left[2p^j R_{jk} q^k\right] . \tag{3.65}
$$

Through the same procedure as above, we can write down a general representation matrix for an arbitrary spatial rotation in 3+1 dimensions

$$
\begin{aligned}
D^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}(R) = {} & \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_0}_{n_0} \frac{\sqrt{n_1! n_2! n_3! m_1! m_2! m_3!}}{(n_1+n_2+n_3)!} \sum_{k_1=0}^{n_1+n_2+n_3} \sum_{k_2=0}^{k_1} \sum_{k_4=0}^{n_2+n_3} \sum_{k_5=0}^{k_4} \\
& \times \binom{n_1+n_2+n_3}{k_1} \binom{k_1}{k_2} \binom{k_2}{n_2+n_3} \binom{n_2+n_3}{k_4} \binom{k_4}{k_5} \binom{k_5}{n_3} \\
& \times \binom{n_3}{m_2+m_3+n_2+2n_3-k_1-k_4} \binom{m_2+m_3+n_2+2n_3-k_1-k_4}{m_3+n_2+2n_3} \\
& \times R_{11}^{n_1+n_2+n_3-k_1} R_{12}^{k_1-k_2} R_{13}^{k_2-n_2-n_3} R_{21}^{n_2+n_3-k_4} \\
& \times R_{22}^{k_4-k_5} R_{23}^{k_5-n_3} R_{32}^{-k1+k_2-k_4+k_5+m_2} R_{33}^{-k_2-k_5+m_3+n_2+2n_3} .
\end{aligned}
\tag{3.66}
$$

For clarity and further uses in different chapters, we provide the representation matrices for rotations of angle θ around the x, y and z axes separately.

$$\begin{aligned}D^{m_0m_1m_2m_3}_{n_0n_1n_2n_3}(\theta\hat{x}) =&\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\sqrt{\frac{n_3!m_2!m_3!}{n_2!}}\times\\ &\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{(-1)^{m_2-k}}{(m_3-n_2+k)!(m_2-k)!}(\cos\theta)^{2k+m_3-n_2}(\sin\theta)^{n_2+m_2-2k}\end{aligned} \tag{3.67}$$

$$\begin{aligned}D^{m_0m_1m_2m_3}_{n_0n_1n_2n_3}(\theta\hat{y}) =&\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_0}_{n_0}\delta^{m_2}_{n_2}\sqrt{\frac{n_1!m_3!m_1!}{n_3!}}\times\\ &\sum_{k=0}^{n_3}\binom{n_3}{k}\frac{(-1)^{m_3-k}}{(m_1-n_3+k)!(m_3-k)!}(\cos\theta)^{2k+m_1-n_3}(\sin\theta)^{n_3+m_3-2k}\end{aligned} \tag{3.68}$$

$$\begin{aligned}D^{m_0m_1m_2m_3}_{n_0n_1n_2n_3}(\theta\hat{z}) =&\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_0}_{n_0}\delta^{m_3}_{n_3}\sqrt{\frac{n_2!m_1!m_2!}{n_1!}}\times\\ &\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{(-1)^{m_1-k}}{(m_2-n_1+k)!(m_1-k)!}(\cos\theta)^{2k+m_2-n_1}(\sin\theta)^{n_1+m_1-2k}.\end{aligned} \tag{3.69}$$

Note that one can easily constrain the representation matrix (3.69) to cover only rotations of Hermite functions on a Euclidean plane by setting $m_0 = n_0 = m_3 = n_3 = 0$, and thus obtain a representation of $SO(2)$.

The matrices (3.67-3.69) are also infinite dimensional, but for spatial rotations, due to the global factor of $\delta^{m_0}_{n_0}$, we find a non-negative invariant number $n = n_1 + n_2 + n_3$. This makes it obvious that the rotation matrices can be reduced to sectors labeled by n, which are finite dimensional in themselves, as there is only a finite number of ways to sum n_1, n_2, n_3 into a non-negative number.

3.2.6 SO(d)

Since the result of the integral (3.65) is written in a way not dependent on the dimension of space, it can be used to represent rotations of any $d-$dimensional Euclidean space, after setting $p_0 = q_0 = 0$. An arbitrary element of $SO(3)$ is given by (3.66) with $m_0 = n_0 = 0$, while an arbitrary element of $SO(2)$ is covered by (3.69) with $m_0 = n_0 = m_3 = n_3 = 0$.

The representation matrices in the case of a general dimension d are given by

$$D^{m_1 \ldots m_d}_{n_1 \ldots n_d} = \delta^{m_1 + \ldots + m_d}_{n_1 + \ldots + n_d} \sqrt{\frac{m_1! \ldots m_d!}{n_1! \ldots n_d!}} \Pi^d_{i,j=1} \sum_{l_{ij}=0}^{l_{i,j+1}} \binom{l_{i,j+1}}{l_{ij}} R_{ij}^{l_{i,j+1}-l_{ij}} q, , \tag{3.70}$$

with

$$\sum_{i=1}^{D} (l_{i,j+1} - l_{ij}) = m_j, \quad l_{i1} = 0, \quad l_{i,d+1} = n_i \tag{3.71}$$

where R_{ij} is an element of $SO(d)$ in the fundamental representation.

3.3 Lorentz Lie algebra in d=4

To find the generators for the Lie algebra, we use the convention

$$D = \exp(K\psi) \tag{3.72}$$

which gives, symbolically

$$K = \frac{\partial D}{\partial \psi}\Big|_{\psi=0}. \tag{3.73}$$

Since by construction the matrices D were unitary, the generators will be antisymmetric

$$K^T = -K\,. \tag{3.74}$$

In the case of boosts, rapidity $\psi = \tanh^{-1}(v)$ is the canonical coordinate for any one-parameter subgroup of boosts in a particular direction of the Lorentz Lie group [60]. Written out for the 4-dimensional case, the three generators of boosts are

$$K_1{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_2}_{n_2} \delta^{m_3}_{n_3} \left(\delta^{m_1}_{n_1+1} \sqrt{(n_0+1)(n_1+1)} - \delta^{m_1}_{n_1-1} \sqrt{n_1 n_0} \right) \tag{3.75}$$

$$K_2{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_1}_{n_1} \delta^{m_3}_{n_3} \left(\delta^{m_2}_{n_2+1} \sqrt{(n_0+1)(n_2+1)} - \delta^{m_2}_{n_2-1} \sqrt{n_2 n_0} \right) \tag{3.76}$$

$$K_3{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_1}_{n_1} \delta^{m_2}_{n_2} \left(\delta^{m_3}_{n_3+1} \sqrt{(n_0+1)(n_3+1)} - \delta^{m_3}_{n_3-1} \sqrt{n_3 n_0} \right) \tag{3.77}$$

The three rotation generators, obtained with the same convention $D = \exp(J\theta)$, are

$$J_1{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_1}_{n_1} \delta^{m_0}_{n_0} \left(\delta^{m_2}_{n_2-1} \sqrt{n_2(n_3+1)} - \delta^{m_2}_{n_2+1} \sqrt{(n_2+1)n_3} \right) \tag{3.78}$$

$$J_2{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_2}_{n_2} \delta^{m_0}_{n_0} \left(\delta^{m_3}_{n_3-1} \sqrt{n_3(n_1+1)} - \delta^{m_3}_{n_3+1} \sqrt{(n_3+1)n_1} \right) \tag{3.79}$$

$$J_3{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_3}_{n_3} \delta^{m_0}_{n_0} \left(\delta^{m_1}_{n_1-1} \sqrt{n_1(n_2+1)} - \delta^{m_1}_{n_1+1} \sqrt{(n_1+1)n_2} \right) \tag{3.80}$$

The Lie algebra satisfies the expected products[3]

$$
\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk} J_k \\
[J_i, K_j] &= \epsilon_{ijk} K_k \\
[K_i, K_j] &= -\epsilon_{ijk} J_k \,.
\end{aligned}
\tag{3.82}
$$

This representation of the Lorentz algebra is not irreducible, and it is a non-trivial problem to reduce it completely. We already noted one level of reducibility, which comes from fixing the number $N = -n_0 + n_1 + n_2 + n_3$. We leave open the question of reducing the representation further.

For future reference, we note the values of the Casimir operators of the Lorentz group $SO(1,3)$ and the rotation group $SO(3)$. There are two Casimir elements of the Lorentz group, both are quadratic

$$
c_1 = \frac{1}{2} M^{\mu\nu} M_{\mu\nu} = \left(\vec{J}^2 - \vec{K}^{\,2}\right) \tag{3.83}
$$

$$
c_2 = -\frac{1}{8} \epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma} = \vec{J} \cdot \vec{K} \,. \tag{3.84}
$$

[3] A more often used convention for the exponential map from the Lie algebra to the Lie group elements is $D = e^{-i\theta J}$, which gives the familiar products

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k
\end{aligned}
\tag{3.81}
$$

In the representation defined above, they become[4]

$$
\begin{aligned}
{c_1}^{m_0m_1m_2m_3}_{n_0n_1n_2n_3} =& \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \times \\
&\Big(-2\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3} a_N \\
&\quad +\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3+2}\sqrt{(n_2-1)n_2(n_3+1)(n_3+2)} \\
&\quad +\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3-2}\sqrt{(n_2+1)(n_2+2)(n_3-1)n_3} \\
&\quad +\delta^{m_0}_{n_0}\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-2}\sqrt{(n_3-1)n_3(n_1+1)(n_1+2)} \\
&\quad +\delta^{m_0}_{n_0}\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3+2}\sqrt{(n_3+1)(n_3+2)(n_1-1)n_1} \\
&\quad +\delta^{m_0}_{n_0}\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3}\sqrt{(n_1-1)n_1(n_2+1)(n_2+2)} \\
&\quad +\delta^{m_0}_{n_0}\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3}\sqrt{(n_1+1)(n_1+2)(n_2-1)n_2} \\
&\quad +\delta^{m_0}_{n_0-2}\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}\sqrt{(n_0-1)n_0(n_1-1)n_1} \\
&\quad +\delta^{m_0}_{n_0+2}\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}\sqrt{(n_0+1)(n_0+2)(n_1+1)(n_1+2)} \\
&\quad +\delta^{m_0}_{n_0-2}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3}\sqrt{(n_0-1)n_0(n_2-1)n_2} \\
&\quad +\delta^{m_0}_{n_0+2}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3}\sqrt{(n_0+1)(n_0+2)(n_2+1)(n_2+2)} \\
&\quad +\delta^{m_0}_{n_0-2}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-2}\sqrt{(n_0-1)n_0(n_3-1)n_3} \\
&\quad +\delta^{m_0}_{n_0+2}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3+2}\sqrt{(n_0+1)(n_0+2)(n_3+1)(n_3+2)}\Big)
\end{aligned} \tag{3.85}
$$

with

$$
a_N = -3(n_1+n_2+n_3) - 2(n_1n_2+n_2n_3+n_3n_1+n_0n_1+n_0n_2+n_0n_3) - 3(1+n_0)\,. \tag{3.86}
$$

[4]Note that the overall factor $\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}$ is redundant in the expressions for the Casimir operators, as this factor's information is already contained in the product of the Kronecker delta's in each term in the bracket.

The second Casimir operator vanishes

$$
\begin{aligned}
{c_2}^{m_0m_1m_2m_3}_{n_0n_1n_2n_3} =&\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\times\\
&\Big(-\delta^{m_0}_{n_0-1}\delta^{m_1}_{n_1-1}\delta^{m_2}_{n_2+1}\delta^{m_3}_{n_3-1}\sqrt{n_0n_1(n_2+1)n_3}\\
&+\delta^{m_0}_{n_0+1}\delta^{m_1}_{n_1+1}\delta^{m_2}_{n_2+1}\delta^{m_3}_{n_3-1}\sqrt{(n_0+1)(n_1+1)(n_2+1)n_3}\\
&+\delta^{m_0}_{n_0-1}\delta^{m_1}_{n_1-1}\delta^{m_2}_{n_2-1}\delta^{m_3}_{n_3+1}\sqrt{n_0n_1n_2(n_3+1)}\\
&-\delta^{m_0}_{n_0+1}\delta^{m_1}_{n_1+1}\delta^{m_2}_{n_2-1}\delta^{m_3}_{n_3+1}\sqrt{(n_0+1)(n_1+1)n_2(n_3+1)}\\
&-\delta^{m_0}_{n_0-1}\delta^{m_2}_{n_2-1}\delta^{m_3}_{n_3+1}\delta^{m_1}_{n_1-1}\sqrt{n_0n_2(n_3+1)n_1}\\
&+\delta^{m_0}_{n_0+1}\delta^{m_2}_{n_2+1}\delta^{m_3}_{n_3+1}\delta^{m_1}_{n_1-1}\sqrt{(n_0+1)(n_2+1)(n_3+1)n_1}\\
&+\delta^{m_0}_{n_0-1}\delta^{m_2}_{n_2-1}\delta^{m_3}_{n_3-1}\delta^{m_1}_{n_1+1}\sqrt{n_0n_2n_3(n_1+1)}\\
&-\delta^{m_0}_{n_0+1}\delta^{m_2}_{n_2+1}\delta^{m_3}_{n_3-1}\delta^{m_1}_{n_1+1}\sqrt{(n_0+1)(n_2+1)n_3(n_1+1)}\\
&-\delta^{m_0}_{n_0-1}\delta^{m_3}_{n_3-1}\delta^{m_1}_{n_1+1}\delta^{m_2}_{n_2-1}\sqrt{n_0n_3(n_1+1)n_2}\\
&+\delta^{m_0}_{n_0+1}\delta^{m_3}_{n_3+1}\delta^{m_1}_{n_1+1}\delta^{m_2}_{n_2-1}\sqrt{(n_0+1)(n_3+1)(n_1+1)n_2}\\
&+\delta^{m_0}_{n_0-1}\delta^{m_3}_{n_3-1}\delta^{m_1}_{n_1-1}\delta^{m_2}_{n_2+1}\sqrt{n_0n_3n_1(n_2+1)}\\
&-\delta^{m_0}_{n_0+1}\delta^{m_3}_{n_3+1}\delta^{m_1}_{n_1-1}\delta^{m_2}_{n_2+1}\sqrt{(n_0+1)(n_3+1)n_1(n_2+1)}\Big)\\
&=0\,. \qquad (3.87)
\end{aligned}
$$

The group $SO(3)$ has a single Casimir element $\vec{J}^2$

$$
\begin{aligned}
(\vec{J})^2{}^{m_0m_1m_2m_3}_{n_0n_1n_2n_3} =&\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_0}_{n_0}\times\\
&\Big(-2\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}(n_1+n_2+n_3+n_1n_2+n_2n_3+n_3n_1)\\
&+\delta^{m_1}_{n_1}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3+2}\sqrt{(n_2-1)n_2(n_3+1)(n_3+2)}\\
&+\delta^{m_1}_{n_1}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3-2}\sqrt{(n_2+1)(n_2+2)(n_3-1)n_3}\\
&+\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-2}\sqrt{(n_3-1)n_3(n_1+1)(n_1+2)}\\
&+\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3+2}\sqrt{(n_3+1)(n_3+2)(n_1-1)n_1}\\
&+\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3}\sqrt{(n_1-1)n_1(n_2+1)(n_2+2)}\\
&+\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3}\sqrt{(n_1+1)(n_1+2)(n_2-1)n_2}\Big)\,. \qquad (3.88)
\end{aligned}
$$

3.3.1 Diagonalization of J_3

Since the conventional way of building the little group for a massless field is by choosing the little momentum to be in the z direction, it is the operator J_3 which can tell us about the helicities of the field. For that reason we explicitly perform its diagonalization. An element of the vector space spanned by Hermite functions

$$\Phi = \sum_{\{m\}=0}^{\infty} p^{m_0 m_1 m_2 m_3} f_{m_0 m_1 m_2 m_3} \tag{3.89}$$

is an eigenvector of the rotation operator if it satisfies the equation

$$J_3 \cdot \Phi = \lambda \Phi \,. \tag{3.90}$$

In terms of components $p^{m_0 m_1 m_2 m_3}$, the eigenvalue equation is

$$J_3{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} p^{n_0 n_1 n_2 n_3} = \lambda p^{m_0 m_1 m_2 m_3} \tag{3.91}$$

where the Einstein convention for summing over repeated indices was used. Since the chosen rotation generator leaves invariant indices n_0, n_3, we suppress them and focus only on the part of interest

$$J_3{}^{m_1 m_2}_{n_1 n_2} = \delta^{m_1+m_2}_{n_1+n_2} \left(\delta^{m_1}_{n_1-1} \sqrt{n_1(n_2+1)} - \delta^{m_1}_{n_1+1} \sqrt{(n_1+1)n_2} \right) \,. \tag{3.92}$$

The eigenvalue equation becomes $J_3{}^{m_1 m_2}_{n_1 n_2} C^{n_1 n_2} = \lambda C^{m_1 m_2}$, or explicitly

$$C^{m_1+1, m_2-1} \sqrt{(m_1+1)m_2} - C^{m_1-1, m_2+1} \sqrt{m_1(m_2+1)} = \lambda C^{m_1, m_2} \,. \tag{3.93}$$

Due to the presence of $\delta^{m_1+m_2}_{n_1+n_2}$ in (3.92), the number $r = n_1 + n_2$ is invariant under the action of the rotation generator, which makes any sector of the vector space with a fixed r finite dimensional. This enables numerical calculations of the coefficients and the eigenvalues, and we report some of them as an explicit example, in the form $C^{m_1 m_2}_{r,\lambda}$ where $r = m_1 + m_2$ and λ is the eigenvalue.

$$C^{m_1 m_2}_{0,0} = \delta^{m_1}_0 \delta^{m_2}_0 \tag{3.94}$$

$$C^{m_1 m_2}_{1,-i} = \frac{1}{\sqrt{2}} (\delta^{m_1}_0 \delta^{m_2}_1 - i \delta^{m_1}_1 \delta^{m_2}_0) \tag{3.95}$$

$$C^{m_1 m_2}_{1,i} = \frac{1}{\sqrt{2}} (\delta^{m_1}_0 \delta^{m_2}_1 + i \delta^{m_1}_1 \delta^{m_2}_0) \tag{3.96}$$

$$C^{m_1 m_2}_{2,0} = \frac{1}{\sqrt{2}} (\delta^{m_1}_0 \delta^{m_2}_2 + \delta^{m_1}_2 \delta^{m_2}_0) \tag{3.97}$$

$$C^{m_1 m_2}_{2,-2i} = \frac{1}{2} (i \delta^{m_1}_0 \delta^{m_2}_2 + \sqrt{2} \delta^{m_1}_1 \delta^{m_2}_1 - i \delta^{m_1}_2 \delta^{m_2}_0) \tag{3.98}$$

$$C^{m_1 m_2}_{2,2i} = \frac{1}{2} (-i \delta^{m_1}_0 \delta^{m_2}_2 + \sqrt{2} \delta^{m_1}_1 \delta^{m_2}_1 + i \delta^{m_1}_2 \delta^{m_2}_0) \tag{3.99}$$

An exact solution to the diagonalization problem is also possible. We can use the following redefinition for the coefficients $C^{m_1 m_2}_{r,\lambda}$ which will enable us to rewrite (3.93) in a simpler way.

$$C^{m_1 m_2}_{r,k} = \frac{(-1)^{(m_2-m_1)/4}}{\sqrt{(2m_1)!!(2m_2)!!}} P^{m_1 m_2}_{r,k} \tag{3.100}$$

where $r = m_1 + m_2$, while k will correspond to the eigenvalue λ, with the exact dependency to be determined. The eigenvalue equation becomes

$$m_1 P^{m_1-1,m_2+1}_{r,k} + m_2 P^{m_1+1,m_2-1}_{r,k} - i\lambda P^{m_1,m_2}_{r,k} = 0 \tag{3.101}$$

Surprisingly, the solution to this equation is given by the Kravchuk matrices [61, 62]. They are defined as

$$K^{(N)}_{ij} = \sum_{k=0}^{N} (-1)^k \binom{j}{k} \binom{N-j}{i-k} \tag{3.102}$$

and we can use them in the following way

$$P^{m_1 m_2}_{r,k} = K^{(r)}_{k\, m_1} = \sum_{i=0}^{r} (-1)^i \binom{m_1}{i} \binom{m_2}{k-i}. \tag{3.103}$$

To prove that this is a solution and to find the eigenvalues, we will first rewrite the Kravchuk matrices in terms of Kravchuk polynomials. Since there is a connection of the Kravchuk polynomials to the hypergeometric function, we will be able to re-express (3.103) using the hypergeometric function. In the last step, we will use known formulas for the hypergeometric function and prove our solution.

The definition of the Kravchuk polynomials is

$$k^{(p)}_n(x,N) = \sum_{i=0}^{n} (-1)^{n-i} \binom{N-x}{n-i} \binom{x}{i} p^{n-i}(1-p)^i \tag{3.104}$$

and they are related to the hypergeometric functions as

$$k^{(p)}_n(x,N) = (-1)^n p^n \binom{N}{n}\ {}_2F_1(-n,-x,-N;1/p)\,. \tag{3.105}$$

To use this in (3.103) we set $x \to m_1$, $N \to m_1 + m_2 = r$, $N - x \to m_2$, $n \to k$ and $p \to 1/2$, from which it follows

$$k^{(1/2)}_k(m_1,r) = \sum_{i=0}^{k} (-1)^{k-i} \binom{m_1}{i} \binom{m_2}{k-i} 2^{-k} \tag{3.106}$$

$$= (-1)^k 2^{-k} K^{(r)}_{k\, m_1} \tag{3.107}$$

$$= (-1)^k 2^{-k} \binom{r}{k}\ {}_2F_1(-k,-m_1,-r;2)\,. \tag{3.108}$$

Therefore

$$P^{m_1,m_2}_{r,k} = \binom{r}{k}\ {}_2F_1(-k,-m_1,-r;2)\,. \tag{3.109}$$

It is most easily seen from the equation above that the integer parameter k can range from 0 to r. We now use the consecutive recurrence relation [63] for the hypergeometric functions :

$$(b-c)\,{}_2F_1(a,b-1,c;z)+(c-2b+(b-a)z)\,{}_2F_1(a,b,c;z) = b(z-1)\,{}_2F_1(a,b+1,c;z)\,. \tag{3.110}$$

To adapt this recurrence relation to our problem, we choose

$$z = 2, \quad a = -k, \quad b = -m_1, \quad c = -r = -m_1 - m_2\,. \tag{3.111}$$

The recurrence relation becomes

$$m_2\ {}_2F_1(-k,-(m_1+1),-r;2)-(r-2k)\ {}_2F_1(-k,-m_1,-r;2) = -m_1\ {}_2F_1(-k,-(m_1-1),-r;2)\,. \tag{3.112}$$

Finally, we can recognize that this is identical to the equation (3.101) with the eigenvalue $\lambda = i(2k-r)$

$$m_1 P^{m_1-1,m_2+1}_{r,k} + m_2 P^{m_1+1,m_2-1}_{r,k} - i\lambda P^{m_1,m_2}_{r,k} = 0\,. \tag{3.113}$$

As we have established above, the parameter k ranges from 0 to r, which means that the eigenvalue λ can for a certain choice of r attain values

$$\lambda_r = -ir, -i(r+1), ..., i(r-1), ir\,. \tag{3.114}$$

The complete solution to the diagonalization problem (3.93) is then given by

$$C^{m_1 m_2}_{r,k} = \frac{(-1)^{(m_2-m_1)/4}}{\sqrt{(2m_1)!!(2m_2)!!}} \sum_{i=0}^{r} (-1)^i \binom{m_1}{i}\binom{m_2}{k-i}\,. \tag{3.115}$$

Chapter 4

Spacetime content and the particle spectrum

In chapter 2 we have defined the MHS gauge field model and shown that the Yang-Mills motivated theory has a perturbatively stable vacuum. The theory was formulated using fields on a master space, and for that reason it is not directly visible what the spacetime content of the model is. To understand the spectrum of the theory in terms of Wigner's classification of elementary particles we need to perform two operations; primarily, we need to find a purely spacetime description for the MHS field, i.e. we need to "integrate out" the auxiliary space dependence and explicitly display the spacetime degrees of freedom. Secondly, we need to extract the free part of the theory, quantize it, and recognize how the Casimir elements of the Poincaré group act on the states created by our linear fields. The latter problem is equivalent to analyzing how the Casimir elements act on-shell on the polarization structure of solutions to the linear equations of motion since these solutions have the same transformation properties as wave functions of a quantum theory.

We will display two perspectives on analyzing the spacetime content of the theory. The first one, a Taylor expansion in the auxiliary space, is conventional and it enables a direct comparison to historical perspectives on higher spin theory, but it is not well suited for our purposes. The second perspective, an expansion in terms of orthogonal functions in the auxiliary space, is novel. We have introduced it for the first time in [1]. It comes out of physical demands, and to understand it we had to develop a new way of representing the Lorentz group, which is done in chapter 3.

With the spacetime content in our hands, we can proceed to the characterization of the particle spectrum. We will present two perspectives here as well. The first one comes from the analysis of the polarization structure of solutions to the linear equations of motion when expanded in terms of Hermite functions. The results from chapter 3 will be indispensable in this perspective, and we will present the obtained results. Such an approach will increase our understanding of the spectrum, but finding all solutions to the posed problems is very involved. A complete characterization will thus ask for an alternative approach. The second perspective comes from analyzing the polarization structure in terms of functions which are solutions to the differential equations posed by the little group generators. Here, in principle, we will be able to write down a complete basis and give an explanation of the particle content of the theory.

4.1 Spacetime fields

4.1.1 Taylor expansion

In the history of exploring higher spin fields, there appeared a perspective of packing a complete tower of higher spin fields into a single structure by using an auxiliary Lorentz vector as a bookkeeping device (see [64] for a review of various such attempts), akin to how we defined the symmetry transformation (2.12) to reproduce (2.2). The resulting object would be of the form

$$h(x,u) = \sum_{n=0}^{\infty} h^{\mu_1 \cdots \mu_n}(x)\, u_{\mu_1} \dots u_{\mu_n} \tag{4.1}$$

where the rank n component fields $h^{\mu_1 \cdots \mu_n}(x)$ were apriori taken to be e.g. Fronsdal fields of spin $s = n$, and the resulting object $h(x,u)$ was considered only a generating function.

Our perspective is different as we take our master field to be a fundamental object of the theory, not only a generating function. Still, by looking at the eq. (2.12), the simplest assumption would be that a spacetime description of the MHS field is furnished by Taylor expanding in the auxiliary space

$$h_a(x,u) = \sum_{n=0}^{\infty} h_a^{(n)\mu_1 \cdots \mu_n}(x)\, u_{\mu_1} \dots u_{\mu_n}\ . \tag{4.2}$$

We deliberately use a Latin index for the master field, and Greek indices for variables of expansion. For reasons mentioned in chapter 2 and which will be examined in more detail in chapter 7, we are motivated to call the Latin index a *frame* index. The coefficients in the expansion are spacetime fields that are Lorentz tensors of rank $n+1$ symmetric in their n (Greek) indices, and which by (2.57) satisfy equations of motion that are of the form

$$\Box h_a^{(n)\mu_1\cdots\mu_n} - \partial_a \partial^b h_b^{(n)\mu_1\cdots\mu_n} + O(h^2) = 0\,. \tag{4.3}$$

From (2.44) we can deduce that the gauge transformations obtained from expanding the gauge parameter as in (2.25) are of the form

$$\delta_\varepsilon h_a^{(n)\mu_1\cdots\mu_n} = \partial_a \varepsilon^{\mu_1\cdots\mu_n} + O(h) \tag{4.4}$$

We see that linearized EoM for spacetime fields defined by (4.2) have the Maxwell form with respect to frame indices, but due to their special role, the fields are not of the type usually considered in the literature (see e.g., [18] for Fronsdal's formulation, [65] for a non-local formulation or [66] for a Maxwell-like formulation). A totally symmetric Lorentz tensor field of rank n satisfying Maxwell-like EoM contains irreducible representations of the Poincaré group with spins $s = n, n-2, n-4, \ldots, 1$ or 0 (see [66]). As the spacetime fields defined by (4.2) have one frame index which is not symmetrized in any way with other (Greek) indices, they presumably propagate additional irreducible Poincaré representations with spins $s \leq n+1$. We shall refer to the spacetime field $h_a^{(n)}$ as the spin-$(n+1)$ field.

The expansion (4.2) leads us to a spacetime description in terms of infinite tower(s) of HS spacetime fields with unbounded spin. If we restrict the MHS potential to an odd function in the auxiliary space (2.60) the tower will only contain spacetime fields with even spin. In [3, 67] it was shown that HS spacetime fields defined by (4.2) linearly couple to the corresponding HS currents when spacetime matter fields are minimally coupled to the MHS potential. We will further elaborate on this in chapter 6. Also, it is straightforward to show that the truncation to $n = 0$ and $n = 1$ sectors is consistent both with the HS transformations and the MHSYM EoM, meaning that on the level of EoM the low-spin sectors ($s = 1$ or 2) may be decoupled from the true HS sector ($s > 2$). These low-spin truncations are analyzed in more detail in chapter 7.

From the form of the HS equations (4.3) admitting the symmetry (4.4) one could be tempted to conclude that the theory has ghosts and, as a consequence, that unitarity is violated. However, such a conclusion is out of reach, since the expansion (4.2) which led to the linearised EoM (4.3) is not suited for the purpose of obtaining a purely spacetime off-shell description. If we substitute (4.2) in the MHSYM action and group the terms by order of u_μ, integrations over the auxiliary space would be divergent at each separate order. As we mention earlier, proper fall-off conditions in the auxiliary space are required, and the *infinite* expansion (4.2) thus has no problems when all (infinite number of terms) are taken together in calculations, but viewing any particular term within the sum by itself, be that in the EoM, action or observables, is ill defined. A consequence is that we cannot obtain regular expressions for classical observables, such as energy and momentum, if only a finite number of spacetime fields $h_a^{(n)\mu_1\ldots\mu_n}(x)$ are non-vanishing, in particular the low-spin ($s \leq 2$) on-shell truncation mentioned above is illusory. The problem with expansion (4.2) is not that it is mathematically incorrect or completely useless, but that the spacetime fields defined by it cannot be treated as independent. If one insists on an expansion like (4.2), the existence of a "hair" consisting of an infinite tail of HS spacetime fields is obligatory in physically acceptable configurations. The main conclusion is that spacetime fields generated by the expansion (4.2) do not correctly reflect the spectrum (particle content) of the MHSYM theory.

4.1.2 Orthogonal functions expansion

An alternative to the Taylor expansion above comes from relaxing the notion of how Lorentz covariance is to be achieved and giving priority to the fact that integrals over the auxiliary space be finite. The latter argument is a reasonable physical demand we put on our theory, and it is most easily seen by looking at the energy of the linearized MHSYM theory

$$U \approx \frac{1}{2g_{\mathrm{ym}}^2} \int d^{d-1}\mathbf{x} \int d^d u \Big(\sum_j F_{0j}(x,u)^2 + \sum_{j<k} F_{jk}(x,u)^2 \Big) < \infty \tag{4.5}$$

To achieve finiteness of the integral above, our master fields need to have appropriate fall-off conditions at large values of the auxiliary space coordinates. We can enforce this condition by using a complete orthonormal set of functions in the auxiliary space $\{f_r(u)\}$ formally indexed by some integer parameter r to expand the MHS potential as

$$h_a(x,u) = \sum_r h_a^{(r)}(x)\, f_r(u) \tag{4.6}$$

where

$$\int d^d u\, f_r(u)\, f_s(u) = \delta_{rs}\ . \tag{4.7}$$

Using such an expansion one arrives at the off-shell space time description with the quadratic part of the Lagrangian given by

$$S_0[h] = -\frac{1}{4g_{\rm ym}^2}\sum_r \int d^dx \left(\partial_a h_b^{(r)} - \partial_b h_a^{(r)}\right)\eta^{ac}\eta^{bd}\delta_{rs}\left(\partial_c h_d^{(s)} - \partial_d h_c^{(s)}\right) . \tag{4.8}$$

On the linear level, the gauge symmetry acts on spacetime fields $h_a^{(r)}(x)$ as

$$\delta_\varepsilon h_a^{(r)}(x) \approx \partial_a \varepsilon^{(r)}(x) \tag{4.9}$$

where $\varepsilon^{(r)}(x)$ are obtained from MHS gauge parameter $\varepsilon(x,u)$ in the same fashion as in (4.6). The above linearized action neither contains dangerous ghosts (of the kind that cannot be removed using gauge freedom), nor runaway modes. In a similar way one can integrate the interacting part of the MHSYM action over the auxiliary space to obtain a purely spacetime action which is a weakly non-local functional of spacetime fields $\{h_a^{(r)}(x)\}$.

The positive definite product of two basis functions in (4.7) might seem in disagreement with the condition of Lorentz covariance since intuition usually leads us to expect the Minkowski metric on the right hand side of equations such as (4.7) if Lorentz covariance is to be achieved. However, as we prove and explicitly construct in chapter 3, we can still achieve Lorentz covariance even in the basis of orthogonal functions, but we have to abandon the notion that the representation will be finite dimensional.

One particularly good choice for the orthonormal basis of functions are multi-dimensional Hermite functions defined in (3.27). We repeat the definition here with modifications due to the fact that the natural auxiliary space variables are u_a, not u^a. $H_n(u)$ are the Hermite polynomials

$$H_n(u) = (-1)^n e^{u^2}\frac{d^n}{du^n}e^{-u^2} , \tag{4.10}$$

where the index n can attain arbitrary non-negative integer values. Hermite functions are defined as

$$f_n(u) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{u^2}{2}} H_n(u) . \tag{4.11}$$

The multidimensional Hermite function that we will use for the expansion in the auxiliary space is defined as

$$f_{n_0\cdots n_{d-1}}(u) = f_{n_0}(u_0)\cdots f_{n_{d-1}}(u_{d-1}) \tag{4.12}$$

They satisfy the orthonormality condition

$$\int du_0 \cdots du_{d-1}\, f_{n_0\cdots n_{d-1}}(u) f_{m_0\cdots m_{d-1}}(u) = \delta_{n_0}^{m_0}\cdots\delta_{n_{d-1}}^{m_{d-1}} \tag{4.13}$$

The MHS potential $h_a(x,u) \equiv h_a(x^b, u_c)$ is now expanded as

$$h_a(x,u) = \sum_{\{n\}=0}^{\infty} h_a^{n_0\cdots n_{d-1}}(x) f_{n_0\cdots n_{d-1}}(u)\,. \tag{4.14}$$

Following the transformation property (2.47), we can deduce the rules for Lorentz transformations of the component fields $h_a^{n_0\cdots n_{d-1}}(x)$. The active transformation is given by

$$h'_a(x,u) = \Lambda_a{}^b h_b(\Lambda^{-1}x, u\Lambda) \tag{4.15}$$

and we can expand both sides of the equation in the Hermite basis

$$\sum_{\{n\}=0}^{\infty} h_a'^{n_0\cdots n_{d-1}}(x) f_{n_0\cdots n_{d-1}}(u) = \Lambda_a{}^b \sum_{\{m\}=0}^{\infty} h_b^{m_0\cdots m_{d-1}}(\Lambda^{-1}x) f_{m_0\cdots m_{d-1}}(u\Lambda)\,. \tag{4.16}$$

Due to (4.13) we can multiply both sides with $f_{r_0\cdots r_d}(u)$, integrate over the auxiliary space, and conclude

$$h_a'^{r_0\cdots r_{d-1}}(x) = \Lambda_a{}^b \sum_{\{m\}=0}^{\infty} L^{r_0\cdots r_{d-1}}_{m_0\cdots m_{d-1}}(\Lambda) h_b^{m_0\cdots m_{d-1}}(\Lambda^{-1}x) \tag{4.17}$$

where

$$L^{r_0\cdots r_{d-1}}_{m_0\cdots m_{d-1}}(\Lambda) = \int du_0 \cdots du_{d-1}\, f_{r_0\cdots r_{d-1}}(u) f_{m_0\cdots m_{d-1}}(u\Lambda) \tag{4.18}$$

is a representation matrix of the Lorentz group in the space of Hermite functions.[1]

[1] We have explicitly constructed the representation matrices in chapter 3, but there is a key difference in (4.18) compared to the formula (3.28) found in the mentioned chapter. Here, the variables of integration are auxiliary space coordinates with lower Lorentz indices, while in chapter 3 we worked with variables with upper Lorentz indices. We now adapt the representation matrices to the current situation. For simplicity, we restrict to two dimensions, and spell out (3.28)

$$D^{m_0 m_1}_{n_0 n_1}(\Lambda) = \int du^0 du^1\, f_{m_0}(u^0) f_{m_1}(u^1) f_{n_0}((\Lambda^{-1}u)^0) f_{n_1}((\Lambda^{-1}u)^1) \tag{4.19}$$

while explicitly in (4.18) we have

$$L^{m_0 m_1}_{n_0 n_1}(\Lambda) = \int du_0 du_1\, f_{m_0}(u_0) f_{m_1}(u_1) f_{n_0}((u\Lambda)_0) f_{n_1}((u\Lambda)_1)\,. \tag{4.20}$$

In the mostly plus signature that we are using we can re-express

$$u_0 = -u^0, \quad u_1 = u^1, \quad (u\Lambda)_0 = -(u\Lambda)^0, \quad (u\Lambda)_1 = (u\Lambda)^1 \tag{4.21}$$

and further we realize

$$(u\Lambda)_\mu = u_\nu \Lambda^\nu{}_\mu, \quad (u\Lambda)^\mu = u_\nu \Lambda^{\nu\mu} = u^\nu \Lambda_\nu{}^\mu = (\Lambda^{-1})^\mu{}_\nu u^\nu = (\Lambda^{-1}u)^\mu\,. \tag{4.22}$$

With the property of Hermite functions $f_n(-u) = (-1)^n f(u)$, we can finally relate

$$L^{m_0 m_1}_{n_0 n_1}(\Lambda) = (-1)^{n_0+m_0} D^{m_0 m_1}_{n_0 n_1}(\Lambda) \tag{4.23}$$

The representation matrices can be found in chapter 3. For further convenience we explicitly write down the generators of the Lorentz group in $d = 4$ in the infinite dimensional representation, adapted to the purposes of this chapter (here we also make the operators Hermitean so the rotation generators J_i differ to (3.75 - 3.80) by a global factor of $-i$, while the boost generators K_i differ by a global factor of i).

$$K_1{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = i\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}\left(\delta^{m_1}_{n_1+1}\sqrt{(n_0+1)(n_1+1)} - \delta^{m_1}_{n_1-1}\sqrt{n_1 n_0}\right) \tag{4.24}$$

$$K_2{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = i\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_1}_{n_1}\delta^{m_3}_{n_3}\left(\delta^{m_2}_{n_2+1}\sqrt{(n_0+1)(n_2+1)} - \delta^{m_2}_{n_2-1}\sqrt{n_2 n_0}\right) \tag{4.25}$$

$$K_3{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = i\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\left(\delta^{m_3}_{n_3+1}\sqrt{(n_0+1)(n_3+1)} - \delta^{m_3}_{n_3-1}\sqrt{n_3 n_0}\right) \tag{4.26}$$

$$J_1{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = i\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_1}_{n_1}\delta^{m_0}_{n_0}\left(\delta^{m_2}_{n_2+1}\sqrt{(n_2+1)n_3} - \delta^{m_2}_{n_2-1}\sqrt{n_2(n_3+1)}\right) \tag{4.27}$$

$$J_2{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = i\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_2}_{n_2}\delta^{m_0}_{n_0}\left(\delta^{m_3}_{n_3+1}\sqrt{(n_3+1)n_1} - \delta^{m_3}_{n_3-1}\sqrt{n_3(n_1+1)}\right) \tag{4.28}$$

$$J_3{}^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} = i\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\delta^{m_3}_{n_3}\delta^{m_0}_{n_0}\left(\delta^{m_1}_{n_1+1}\sqrt{(n_1+1)n_2} - \delta^{m_1}_{n_1-1}\sqrt{n_1(n_2+1)}\right) \tag{4.29}$$

4.1.3 Linear solutions and helicity

In this subsection we focus on the particular number of dimensions $d = 4$. We can use the expansion (4.14) and insert it into linearized EoM obtained from (2.57). The component fields in the expansion satisfy

$$\Box h_a^{n_0 n_1 n_2 n_3}(x) - \partial_a \partial^b h_b^{n_0 n_1 n_2 n_3}(x) = 0 \tag{4.30}$$

and they enjoy a linearized gauge symmetry of the form

$$\delta_\varepsilon h_a^{n_0 n_1 n_2 n_3}(x) = \partial_a \varepsilon^{n_0 n_1 n_2 n_3}(x)\,. \tag{4.31}$$

To find out about the helicity of the field, we can write down a plane wave solution to the EoM (4.30), use the freedom available through (4.31) to fix the gauge and choose a direction of propagation (conventionally, we choose the z-direction). Such a solution is given by

$$h^{a\, n_0 n_1 n_2 n_3}(x) = \epsilon^a_{(\pm)}\, p^{n_0 n_1 n_2 n_3} e^{ikx} \tag{4.32}$$

The same prefactor $(-1)^{n_0+m_0}$ is present in any number of dimensions.

where $k^2 = 0$, meaning that the field is massless,

$$\epsilon^a_{(\pm)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \tag{4.33}$$

and $p^{n_0 n_1 n_2 n_3}$ is an apriori unconstrained polarization factor in the infinite dimensional unitary representation of the Lorentz group.

The helicity of a plane wave can be calculated as the eigenvalue of the rotation generator around the axis of propagation. As we have followed the convention and chosen the z axis as the axis of propagation, we want to find the eigenvalue of the rotation operator J_3. When acting on (4.32), which is in a mixed representation, each generator will have two parts; one belonging to the finite dimensional representation (indices a, b), and one belonging to the infinite dimensional representation (indices $n_0, ..., n_3$), i.e.

$$(J_3)^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}{}^{a}{}_{b} = (J_3)^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}\,\delta^a_b + \delta^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}\,(J_3)^a{}_b \tag{4.34}$$

where $(J_3)^a{}_b$ is in the fundamental (vector) representation of the Lorentz group, given explicitly by

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.35}$$

It is then easy to see that (Einstein summation convention assumed)

$$(J_3)^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}{}^{a}{}_{b}\,\epsilon^b_{(\pm)}\,p^{n_0 n_1 n_2 n_3} = (\pm 1 + \lambda)\epsilon^a_{(\pm)}\,p^{m_0 m_1 m_2 m_3} \tag{4.36}$$

where

$$p^{n_0 n_1 n_2 n_3} = d^{n_0 n_3} C^{n_1 n_2}_{r,k} \tag{4.37}$$

with $d^{n_0 n_3}$ arbitrary, $C^{n_1 n_2}_{r,k}$ given by (3.115), and $\lambda = (2k - r)$. As explained in chapter 3,

$$r = n_1 + n_2, \quad k = 0, 1, ..., r\,.$$

All together, this means that the plane waves (4.32) can carry helicity $\pm(1 + n_1 + n_2), \pm(n_1 + n_2), \pm(n_1 + n_2 - 1), \cdots \pm 1(0)$, depending on $n_1 + n_2$ being even or odd. We can also notice that a single value of helicity can appear in infinitely many polarization factors.

Another approach leading to the same conclusion comes from using a complete orthonormal basis in the auxiliary space built over spherical harmonics $\{g_{r_0 nlm}(u)\}$ which are by construction diagonal in the rotation generator around the axis of propagation e.g.,

$$g_{r_0 nlm}(\bar{u}, \hat{\mathbf{z}}) = f_{r_0}(\bar{u}_0)\, F_n(|\bar{\mathbf{u}}|)\, Y_l^m(\theta, \phi) \tag{4.38}$$

where F_n are Laguerre functions, Y_l^m are spherical harmonics, and $r_0 = 0, 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, $l = n, n-2, \ldots, 1(0)$, $m = -l, -l+1, \ldots, l$. Then the plane wave solutions for the MHS potential can be expanded in this basis as

$$\boldsymbol{\epsilon}_{\sigma r_0 nlm}(\mathbf{k})\, e^{ik\cdot x}\, g_{r_0 nlm}(\bar{u}, \hat{\mathbf{k}}) \quad , \quad \mathbf{k} \cdot \boldsymbol{\epsilon}_{\sigma r_0 nlm}(\mathbf{k}) = 0 \quad , \quad k^2 = 0 \tag{4.39}$$

with $\sigma = \pm 1$. Helicity is given by $\sigma + m$, which shows that there is an infinite number of fluctuations for every value of the helicity.

4.2 Wigner's classification

By the principle of relativity, isometries of a spacetime are symmetries of a physical system. Wigner's classification of elementary particles [13] is thus a classification of the isometry group (in our case, the Poincaré group) represented on the space of one particle states. Here we would like to display the basics of Wigner's method and highlight the relationship to the plane wave solutions of the linear equations of motion. This exposition closely follows the first volume of Weinberg's Quantum Theory of Fields [68] while additional details can be found in [69, 70, 71]

The Lie algebra of the Poincaré group consists of 10 generators, whose Lie brackets are given as:

$$[P_\mu, P_\nu] = 0 \tag{4.40}$$

$$i[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \tag{4.41}$$

$$i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \tag{4.42}$$

Building a representation of the Poincaré group on a space of one-particle states can be done by the method of induced representations. In $d = 4$, the Poincaré group has a quadratic and a quartic Casimir element

$$C_2 = -P^\mu P_\mu = -P^2, \quad C_4 = W^\mu W_\mu = W^2 \tag{4.43}$$

where the translation generators are P^μ, with the Pauli-Lubanski vector W^μ defined as

$$W_\rho = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} P^\sigma \,, \tag{4.44}$$

and the Lorentz generators denoted by $M^{\mu\nu}$. The orbital part of the Lorentz generators ($\sim X^\mu P^\nu - X^\nu P^\mu$) does not contribute to the Pauli-Lubanski vector.

It is natural to label the single particle states with the values of the Casimir elements. Further, since momentum operators form an abelian subgroup, we work with their eigenvectors and label them as

$$|p^2, \mu^2, p, \sigma\rangle \tag{4.45}$$

such that

$$P^2 |p^2, \mu^2, p, \sigma\rangle = p^2 |p^2, \mu^2, p, \sigma\rangle \,, \quad W^2 |p^2, \mu^2, p, \sigma\rangle = \mu^2 |p^2, \mu^2, p, \sigma\rangle \tag{4.46}$$

and

$$P^\mu |p^2, \mu^2, p, \sigma\rangle = p^\mu |p^2, \mu^2, p, \sigma\rangle \tag{4.47}$$

with σ labeling other degrees of freedom which are to be determined. For each value of p^2 we can choose a standard momentum k^μ such that any other p^μ with the same value of p^2 can be obtained by a Lorentz transformation

$$p^\mu = S(p)^\mu{}_\nu k^\nu \,. \tag{4.48}$$

For each such choice of k^μ there is a set of Lorentz transformations leaving k^μ invariant

$$B^\mu{}_\nu k^\nu = k^\mu \tag{4.49}$$

which form a subgroup of the Lorentz group named the little group corresponding to the standard momentum k^μ. It can be shown [69, 72] that the generators of the little group are given by the components of the Pauli-Lubanski vector with the specific standard momentum

$$W_\rho = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} k^\sigma \,. \tag{4.50}$$

Since $W_\mu k^\mu = 0$, we know that the Lie algebra of any little group in $d = 4$ will be three-dimensional. The possibilities are given in table 4.1, and we are primarily intersted in the positive energy ($p^0 > 0$) options.

$C_2 = -p^2$	Standard momentum k^μ	Little group	Example
$p^2 = -m^2$	$(\pm m, 0, 0, 0)$	$SO(3)$	$W^\pm$ and Z bosons
$p^2 = 0$	$(\pm\omega, 0, 0, \omega)$	$ISO(2)$	photons
$p^2 = n^2$	$(0, 0, 0, n)$	$SO(2,1)^\uparrow$	tachyons
$p^2 = 0$	$(0, 0, 0, 0)$	$SO(3,1)^\uparrow$	vacuum

Table 4.1: Representations of the Poincaré group

While translations act on the basis vectors as

$$U(a)\,|p^2, w^2, p, \sigma\rangle = e^{-ipa}\,|p^2, w^2, p, \sigma\rangle \ , \tag{4.51}$$

it can be shown that homogeneous Lorentz transformation act as

$$U(\Lambda)\,|p^2, w^2, p, \sigma\rangle = \mathcal{N}\sum_{\sigma'}\mathcal{D}_{\sigma'\sigma}(W(\Lambda, p))\,|p^2, w^2, \Lambda p, \sigma'\rangle \ , \tag{4.52}$$

where $\mathcal{N}$ is a normalization factor and $W(\Lambda, p) = S^{-1}(\Lambda p)\Lambda S(p)$ is an element of the little group for a particular standard momentum. Matrices $\mathcal{D}_{\sigma'\sigma}(W)$ furnish an irreducible representation of the little group, and their construction is then sufficient to properly characterize the one particle state. For each choice of standard momentum, the quartic Casimir of the Poincaré group is equal to the Casimir operator of the little group. We will be especially interested in the massless case, and for that reason we focus in more detail on the group $ISO(2)$.

The Lorentz generators $M^{\mu\nu}$ can be unpacked so that the generators of rotations J_i and boosts K_i can individually be recognized as

$$J_1 = M^{23}, \quad J_2 = M^{31}, \quad J_3 = M^{12} \tag{4.53}$$

$$K_1 = M^{10}, \quad K_2 = M^{20}, \quad K_3 = M^{30} \tag{4.54}$$

If we choose the standard momentum as $k^\mu = (\omega, 0, 0, \omega)$, we can explicitly find the components of the Pauli-Lubanski vector which are the generators of the little group for the case of massless particles

$$W^\mu = \omega(J_3, J_1 - K_2, J_2 + K_1, J_3)\,. \tag{4.55}$$

It is convenient to name the generators

$$A = \omega(J_1 - K_2), \quad B = \omega(J_2 + K_1)\,. \tag{4.56}$$

By knowing the commutation relations of the Lorentz algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \tag{4.57}$$

it is easy to check that A, B and J_3 span the Lie Algebra $\mathfrak{iso}(2)$

$$[A, B] = 0, \quad [J_3, A] = iB, \quad [J_3, B] = -iA\,. \tag{4.58}$$

The quartic Casimir in this choice of standard momentum is then given by

$$W_\mu W^\mu \equiv W^2 = \omega^2 (J_1 - K_2)^2 + \omega^2 (J_2 + K_1)^2 \tag{4.59}$$

$$= (A^2 + B^2) \tag{4.60}$$

and naturally it commutes with all elements of the little group algebra

$$[W^2, A] = [W^2, B] = [W^2, J_3] = 0\,. \tag{4.61}$$

In appendix A.2 we report on how to build a unitary representation of the Lie algebra $\mathfrak{iso}(2)$, and the Lie group $ISO(2)$. Here we point out its main features.

The faithful irreducible unitary representations of the little group $ISO(2)$ which have a non-vanishing value of the Casimir operator W^2 are necessarily infinite dimensional. If written in a basis diagonal in the rotation operator around the standard momentum, it can be seen that each irreducible representation contains an infinite tower of helicity states mixing under Lorentz transformations. For that reason, such representations are usually named "infinite-spin". A different basis choice is possible, as we spell out in the appendix A.2, which motivates a different name - "continuous spin". This class of representations was originally considered by Wigner to be unsuitable for a physical use, since the infinite tower of helicities would have to correspond to an infinite heat capacity. However in the recent years there has been a revived interest for this class of particles and in analyzing their kinematic and dynamical aspects with more scrutiny ([73, 74, 75, 76, 77, 78]).

There is also a possibility of a non-faithful representation of the little group $ISO(2)$ where the operators A, B act trivially. In this case W^2 gives a vanishing value and the little group becomes isomorphic to $SO(2)$. The representations are one-dimensional, with the only non-trivial operator being the rotation around the standard momentum. The eigenvectors of this rotation are the ordinary helicity states describing particles corresponding to massless fields of a fixed spin such as the Maxwell field, linear Einstein gravity, higher spin fields of the Fronsdal type, etc.

There is an important relationship between the matrices $\mathcal{D}_{\sigma'\sigma}(W)$ which, as we have seen, act on the one particle states, and the Lorentz transformation matrices we use to express quantum field components in different inertial frames. From the creation and annihilation operators we can build a quantum field, where r stands for any set of Lorentz indices

$$h^r(x) = \sum_\sigma \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}} \left(u^r(\mathbf{p},\sigma)a(\mathbf{p},\sigma)e^{-ipx} + v^r(\mathbf{p},\sigma)a^\dagger(\mathbf{p},\sigma)e^{ipx}\right) . \tag{4.62}$$

Under a Lorentz transformation it behaves as

$$U(\Lambda)h^r(x)U^{-1}(\Lambda) = \sum_s L(\Lambda^{-1})^{rs} h^s(\Lambda x) \tag{4.63}$$

where $L(\Lambda)$ is a representation of the Lorentz group (finite or infinite-dimensional) on the space of fields. This equation can be seen as a compatibility condition [70, 68, 72] between the infinite-dimensional unitary Fock-space representation on the one particle states and the representation of the homogeneous Lorentz group on the space of fields. In a straightforward manner, it can be brought down to compatibility equations for the polarization functions in the standard momentum $\mathbf{k}$

$$\sum_{\sigma'} u^r(\mathbf{k},\sigma')\mathcal{D}_{\sigma'\sigma}(W^{-1}) = \sum_s L(W^{-1})^{rs} u^s(\mathbf{k},\sigma) \tag{4.64}$$

$$\sum_{\sigma'} v^r(\mathbf{k},\sigma')\mathcal{D}^*_{\sigma'\sigma}(W^{-1}) = \sum_s L(W^{-1})^{rs} v^s(\mathbf{k},\sigma) . \tag{4.65}$$

On the level of the algebra of the little group we have

$$\sum_{\sigma'} u^r(\mathbf{k},\sigma')\mathcal{J}_{\sigma'\sigma} = \sum_s J^{rs} u^s(\mathbf{k},\sigma) \tag{4.66}$$

$$\sum_{\sigma'} v^r(\mathbf{k},\sigma')\mathcal{J}^*_{\sigma'\sigma} = -\sum_s J^{rs} v^s(\mathbf{k},\sigma) . \tag{4.67}$$

which follows from the expansion $\mathcal{D}_{\sigma'\sigma} \approx \delta_{\sigma\sigma'} + i\theta\mathcal{J}_{\sigma\sigma'}$, and $L^{rs} \approx \delta^{rs} + i\theta J^{rs}$.

The polarization functions in the standard momentum thus carry the representation of the little group and contain also the information about the quartic Casimir operator. By classifying polarization functions of a certain field, through solving the eigensystem

$$\sum_s L\left(W^2\right)^{rs} u^s(\mathbf{k},\sigma) = \mu^2 u^r(\mathbf{k},\sigma) , \tag{4.68}$$

where μ^2 on the right hand side is constant due to (4.46), we can learn about supported particle types, i.e values of the Casimir operator W^2, associated with that particular field.

4.3 The quartic Casimir in the Hermite expansion

Our first goal is to build an explicit expression for (4.60) for the case of the plane wave solution (4.32), and then attempt to find possible eigenvalues. For the sake of the clarity of argument, we will demonstrate the procedure on the Maxwell field before following these steps for the MHSYM component field.

In the case of electrodynamics, a plane-wave solution of the equations

$$\Box A^{\mu} - \partial^{\mu}\partial \cdot A = 0 \tag{4.69}$$

with momentum oriented in the z direction $k^{\mu} = (\omega, 0, 0, \omega)$ is given by

$$A^{\mu}(x) = \epsilon^{\mu} e^{ikx} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} e^{ikx} \, . \tag{4.70}$$

Since Maxwell's field is a Lorentz vector, we can explicitly use the vector representation of the Lorentz generators

$$\left(J^{\mu\nu}\right)^{\alpha}{}_{\beta} = i\left(\eta^{\mu\alpha}\delta^{\nu}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}_{\beta}\right) \, . \tag{4.71}$$

Using the identifications (4.53) and the result (4.60) the quartic Casimir element is explicitly given by

$$W^2 = \omega^2 \begin{pmatrix} -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix} \tag{4.72}$$

so it is readily visible through application of (4.68) that there is a single eigenvalue of the quartic Casimir, and it is vanishing

$$\left(W^2\right)^{\alpha}{}_{\beta} \cdot A^{\beta} = 0 \, .$$

Quartic Casimir of the on-shell MHS field

As already stated, the MHS field is in a mixed representation of the Lorentz group - a direct product of the finite dimensional vector representation and the infinite dimensional unitary representation. The generators will be a direct sum of two parts; one belonging to the finite dimensional representation (indices a, b), and one belonging to the infinite dimensional representation (indices n_0, n_1, n_2, n_3 in case of the Hermite expansion in $d = 4$), e.g.

$$(J_1)_{n_0n_1n_2n_3}^{m_0m_1m_2m_3}{}^{a}{}_{b} = (J_1)_{n_0n_1n_2n_3}^{m_0m_1m_2m_3}\,\delta_b^a + \delta_{n_0n_1n_2n_3}^{m_0m_1m_2m_3}\,(J_1)^a{}_b \tag{4.73}$$

A compactified notation uses a capital $\mathbf{N}$ instead of the tuple $\{n_0n_1n_2n_3\}$

$$(J_1)_{\mathbf{N}}^{\mathbf{M}}{}^{a}{}_{b} = (J_1)_{\mathbf{N}}^{\mathbf{M}}\,\delta_b^a + \delta_{\mathbf{N}}^{\mathbf{M}}\,(J_1)^a{}_b\,. \tag{4.74}$$

The Casimir element (4.60) is then given by (repeated indices summed over)

$$\begin{aligned}\left(W^2\right)_{\mathbf{N}}^{\mathbf{M}}{}^{a}{}_{c} =&\omega^2\Big(J_1{}^a{}_bJ_1{}^b{}_c + J_2{}^a{}_bJ_2{}^b{}_c + K_1{}^a{}_bK_1{}^b{}_c + K_2{}^a{}_bK_2{}^b{}_c - 2J_1{}^a{}_bK_2{}^b{}_c + 2K_1{}^a{}_bJ_2{}^b{}_c\Big)\delta_{\mathbf{N}}^{\mathbf{M}}\\ &+\omega^2\Big(J_1{}_{\mathbf{R}}^{\mathbf{M}}J_1{}_{\mathbf{N}}^{\mathbf{R}} + J_2{}_{\mathbf{R}}^{\mathbf{M}}J_2{}_{\mathbf{N}}^{\mathbf{R}} + K_1{}_{\mathbf{R}}^{\mathbf{M}}K_1{}_{\mathbf{N}}^{\mathbf{R}} + K_2{}_{\mathbf{R}}^{\mathbf{M}}K_2{}_{\mathbf{N}}^{\mathbf{R}} - 2J_1{}_{\mathbf{R}}^{\mathbf{M}}K_2{}_{\mathbf{N}}^{\mathbf{R}} + 2K_1{}_{\mathbf{R}}^{\mathbf{M}}J_2{}_{\mathbf{N}}^{\mathbf{R}}\Big)\delta_c^a\\ &+\omega^2\Big(2J_1{}^a{}_cJ_1{}_{\mathbf{N}}^{\mathbf{M}} + 2J_2{}^a{}_cJ_2{}_{\mathbf{N}}^{\mathbf{M}} + 2K_1{}^a{}_cK_1{}_{\mathbf{N}}^{\mathbf{M}} + 2K_2{}^a{}_cK_2{}_{\mathbf{N}}^{\mathbf{M}}\\ &-2K_2{}^a{}_cJ_1{}_{\mathbf{N}}^{\mathbf{M}} - 2K_2{}_{\mathbf{N}}^{\mathbf{M}}J_1{}^a{}_c + 2K_1{}^a{}_cJ_2{}_{\mathbf{N}}^{\mathbf{M}} + 2K_1{}_{\mathbf{N}}^{\mathbf{M}}J_2{}^a{}_c\Big)q,\,. \end{aligned}\tag{4.75}$$

The first bracket contains the finite-dimensional vector representation of the W^2, and it is multiplied by $\delta_{\mathbf{N}}^{\mathbf{M}}$. As in the case of a finite dimensional massless vector field, this will give a 0 when acting on the polarization vector ϵ^a found in (4.32).

The mixed contributions to the Casimir can be rewritten as

$$(W^2_{mixed})_{\mathbf{N}}^{\mathbf{M}a}{}_c = 2A^a{}_cA_{\mathbf{N}}^{\mathbf{M}} + 2B^a{}_cB_{\mathbf{N}}^{\mathbf{M}}$$

where A, B were defined in (4.56). When $A^a{}_c$ or $B^a{}_c$ act on the polarization vector ϵ^a, the result will be proportional to the standard momentum, e.g for $A^a{}_c$

$$i\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix} = \frac{\pm 1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\propto k^a \tag{4.76}$$

i.e. a pure gauge contribution in the finite-dimensional sector. A non-trivial eigenvalue of the quartic Casimir for the MHS field can thus come only from the second line of (4.75), which contains the infinite-dimensional part

$$\left(W^2_{inf}\right)_{\mathbf{N}}^{\mathbf{M}}\delta^a{}_c = \omega^2\delta_c^a(J_1{}_{\mathbf{R}}^{\mathbf{M}}J_1{}_{\mathbf{N}}^{\mathbf{R}} + J_2{}_{\mathbf{R}}^{\mathbf{M}}J_2{}_{\mathbf{N}}^{\mathbf{R}} + K_1{}_{\mathbf{R}}^{\mathbf{M}}K_1{}_{\mathbf{N}}^{\mathbf{R}} + K_2{}_{\mathbf{R}}^{\mathbf{M}}K_2{}_{\mathbf{N}}^{\mathbf{R}} - 2K_2{}_{\mathbf{R}}^{\mathbf{M}}J_1{}_{\mathbf{N}}^{\mathbf{R}} + 2J_2{}_{\mathbf{R}}^{\mathbf{M}}K_1{}_{\mathbf{N}}^{\mathbf{R}})\,. \tag{4.77}$$

We now use the explicit expressions for the infinite-dimensional generators (4.24-4.29), and arrive at the result written out without the use of the compactified notation.

$$\begin{aligned}
\left(W^2_{inf}\right)^{m_0m_1m_2m_3}_{n_0n_1n_2n_3} =&\omega^2\delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3}\times \\
&\Bigg(2\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}(1+n_0+n_3)(1+n_1+n_2) \\
&-\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3-2}\sqrt{(n_2+1)(n_2+2)(n_3-1)n_3} \\
&-\delta^{m_0}_{n_0}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3+2}\sqrt{(n_2-1)n_2(n_3+2)(n_3+1)} \\
&-\delta^{m_0}_{n_0}\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3+2}\sqrt{(n_1-1)n_1(n_3+1)(n_3+2)} \\
&-\delta^{m_0}_{n_0}\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-2}\sqrt{(n_1+1)(n_1+2)(n_3-1)n_3} \\
&-\delta^{m_0}_{n_0+2}\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}\sqrt{(n_0+1)(n_0+2)(n_1+1)(n_1+2)} \\
&-\delta^{m_0}_{n_0-2}\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3}\sqrt{(n_0-1)n_0(n_1-1)n_1} \\
&-\delta^{m_0}_{n_0+2}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3}\sqrt{(n_0+1)(n_0+2)(n_2+2)(n_2+1)} \\
&-\delta^{m_0}_{n_0-2}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3}\sqrt{(n_0-1)n_0(n_2-1)n_2} \\
&+2\delta^{m_0}_{n_0+1}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2+2}\delta^{m_3}_{n_3-1}\sqrt{(n_0+1)(n_2+1)(n_2+2)n_3} \\
&-2\delta^{m_0}_{n_0+1}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3+1}(\sqrt{(n_0+1)n_2n_2(n_3+1)}+\sqrt{(n_0+1)(n_1+1)(n_1+1)(n_3+1)}) \\
&-2\delta^{m_0}_{n_0-1}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-1}(\sqrt{n_0(n_2+1)(n_2+1)n_3}+\sqrt{n_0n_1n_1n_3}) \\
&+2\delta^{m_0}_{n_0-1}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2-2}\delta^{m_3}_{n_3+1}\sqrt{n_0(n_2-1)n_2(n_3+1)} \\
&+2\delta^{m_0}_{n_0+1}\delta^{m_1}_{n_1+2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-1}\sqrt{(n_0+1)(n_1+1)(n_1+2)n_3} \\
&+2\delta^{m_0}_{n_0-1}\delta^{m_1}_{n_1-2}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3+1}\sqrt{n_0(n_1-1)n_1(n_3+1)}\Bigg). \qquad (4.78)
\end{aligned}$$

When acting on the field polarization factors $p^{n_0 n_1 n_2 n_3}$ in (4.32), we get

$$
\begin{aligned}
&\left(W^2_{inf}\right)^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} p^{n_0 n_1 n_2 n_3} = \\
&\quad 2p^{m_0 m_1 m_2 m_3}\left[(1+m_0+m_3)(1+m_1+m_2)\right] \\
&\quad -p^{m_0 m_1 (m_2-2)(m_3+2)}\sqrt{(m_2-1)m_2(m_3+1)(m_3+2)} \\
&\quad -p^{m_0 m_1 (m_2+2)(m_3-2)}\sqrt{(m_2+1)(m_2+2)m_3(m_3-1)} \\
&\quad -p^{m_0 (m_1+2) m_2 (m_3-2)}\sqrt{(m_1+1)(m_1+2)(m_3-1)m_3} \\
&\quad -p^{m_0 (m_1-2) m_2 (m_3+2)}\sqrt{(m_1-1)m_1(m_3+1)(m_3+2)} \\
&\quad -p^{(m_0-2)(m_1-2) m_2 m_3}\sqrt{(m_0-1)m_0(m_1-1)m_1} \\
&\quad -p^{(m_0+2)(m_1+2) m_2 m_3}\sqrt{(m_0+1)(m_0+2)(m_1+1)(m_1+2)} \\
&\quad -p^{(m_0-2) m_1 (m_2-2) m_3}\sqrt{(m_0-1)m_0 m_2(m_2-1)} \\
&\quad -p^{(m_0+2) m_1 (m_2+2) m_3}\sqrt{(m_0+1)(m_0+2)(m_2+1)(m_2+2)} \\
&\quad +2p^{(m_0-1) m_1 (m_2-2)(m_3+1)}\sqrt{m_0(m_2-1)m_2(m_3+1)} \\
&\quad -2p^{(m_0-1) m_1 m_2 (m_3-1)}\sqrt{m_0 m_2 m_2 m_3)} \\
&\quad -2p^{(m_0+1) m_1 m_2 (m_3+1)}\sqrt{(m_0+1)(m_2+1)(m_2+1)(m_3+1)} \\
&\quad +2p^{(m_0+1) m_1 (m_2+2)(m_3-1)}\sqrt{(m_0+1)(m_2+1)(m_2+2)m_3} \\
&\quad -2p^{(m_0-1) m_1 m_2 (m_3-1)}\sqrt{m_0(m_1+1)(m_1+1)m_3} \\
&\quad +2p^{(m_0-1)(m_1-2) m_2 (m_3+1)}\sqrt{m_0(m_1-1)m_1(m_3+1)} \\
&\quad +2p^{(m_0+1)(m_1+2) m_2 (m_3-1)}\sqrt{(m_0+1)(m_1+1)(m_1+2)m_3} \\
&\quad -2p^{(m_0+1) m_1 m_2 (m_3+1)}\sqrt{(m_0+1)m_1 m_1(m_3+1)}\,.
\end{aligned}
\tag{4.79}
$$

4.3.1 Casimir eigenvalue problem

To learn about the particle spectrum of our theory, following (4.68), we should solve the eigensystem

$$
\left(W^2_{inf}\right)^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3} p^{n_0 n_1 n_2 n_3} = \mu^2 p^{m_0 m_1 m_2 m_3}\,. \tag{4.80}
$$

Even before explicitly trying to find eigenvectors and eigenvalues in (4.80), we can conclude from (4.79) that there will exist non-trivial states, i.e. the expression (4.79) shows that a polarization factor $p^{n_0 n_1 n_2 n_3}$ used in (4.32) will in general not give a vanishing eigenvalue, through which we can confirm that the MHS formalism supports a description of the infinite spin particles.

One way of tackling the eigenvalue problem is by computer assisted iterative solving, which could give us a hint for an appropriate ansatz. It can be seen by the structure of the Casimir that any eigenvector should necessarily have an infinite number of components[2], so we could only hope for hints coming from a truncated calculation. The Casimir operator can be rewritten in a basis of eigenvectors of J_3 which were found in chapter 3, but the complexity of the problem remains. So far, we are left with finding educated guesses and one of them comes from the "massless limit" of eigenvectors of J^2.

Massless limit of massive states

In the case of massive particles, the little group is $SO(3)$, and the Casimir operator is simply $\vec{J}^2$. An appropriately performed Inönü-Wigner contraction of a representation of $SO(3)$ can give us a representation of $ISO(2)$, which is the little group in the case of massless particles. The limiting procedure entails the limits $m \to 0, v \to 1$ while keeping fixed $m\gamma = \frac{m}{\sqrt{1-v^2}} = \omega$.

The simplest simultaneous eigenvector of $\vec{J}^2$ (3.88) and J_3 (4.29, 3.80) in the representation over Hermite functions is

$$\Phi_{n=0,s=0,\lambda=0}(u) = \delta_0^{n_0}\delta_0^{n_1}\delta_0^{n_2}\delta_0^{n_3} f_{n_0 n_1 n_2 n_3}(u) \tag{4.81}$$

where $n = n_1 + n_2 + n_3$ and s corresponds to the eigenvalue of $\vec{J}^2 = s(s+1)$ while λ is an eigenvalue of J_3. We can boost (4.81) with velocity v in the z direction to prepare it for the massless limit. The transformation matrices for a finite boost are (3.58, 4.23)

$$\begin{aligned} D^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}(v) = {} & \sqrt{\frac{m_3! n_0!}{n_3! m_0!}} \delta^{-m_0+m_1+m_2+m_3}_{-n_0+n_1+n_2+n_3} \delta^{m_1}_{n_1} \delta^{m_2}_{n_2} \times \\ & \sum_{j=0}^{m_0} \binom{m_0}{j} \binom{n_3}{m_3 - j} (-1)^j \sqrt{1-v^2}^{m_3+m_0+1-2j} v^{2j-m_3+n_3} . \end{aligned} \tag{4.82}$$

Boosting the chosen eigenvector we get

$$\sum_{n_0,n_1,n_2,n_3=0}^{\infty} D^{m_0 m_1 m_2 m_3}_{n_0 n_1 n_2 n_3}(v) \cdot \delta_0^{n_0}\delta_0^{n_1}\delta_0^{n_2}\delta_0^{n_3} = \delta_0^{m_1}\delta_0^{m_2}\delta_{m_3}^{m_0}\sqrt{1-v^2}(-v)^{m_3} . \tag{4.83}$$

Since $m\gamma \to \omega$ while $v \to 1$, the limiting procedure gives us

$$p^{n_0 n_1 n_2 n_3} \to \delta_0^{m_1}\delta_0^{m_2}\delta^{m_0,m_3}(-1)^{m_3} . \tag{4.84}$$

[2] Terms such as $\delta^{m_0}_{n_0-1}\delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\delta^{m_3}_{n_3-1}$ will always simultaneously raise the values of the 0th and 3rd index, thus a closed solution cannot have a finite number of terms.

A more general case could be an eigenvector of J_3 and J^2 of the form

$$\Phi_{n=n,s=n,\lambda=n}(u) = z^{n_0} C_{n,n}^{n_1 n_2} \delta_0^{n_3} f_{n_0 n_1 n_2 n_3}(u)\,. \tag{4.85}$$

In the sector $n_1 + n_2 + n_3 = n$ where $s = n$ and $\lambda = n$, there is only one such vector, and the factor $C_{n,n}^{n_1 n_2}$ is given in (3.115). The index n_3 has to be equal to 0, while n_0 is arbitrary (or fixed by a choice of $N = -n_0 + n$), meaning that the factor z^{n_0} must have the form

$$z^{n_0} = const. \cdot \delta_{n-N}^{n_0}\,. \tag{4.86}$$

We boost (4.85) in the z direction to prepare it for the massless limit.

$$D_{n_0 n_1 n_2 n_3}^{m_0 m_1 m_2 m_3}(v) z^{n_0} C_{n,n}^{n_1 n_2} \delta_0^{n_3} = \sqrt{\frac{m_0!}{m_3!(m_0-m_3)!}} z^{m_0-m_3} C_{m_1+m_2,m_1+m_2}^{m_1 m_2} \sqrt{1-v^2}^{m_0-m_3+1} (-v)^{m_3} \tag{4.87}$$

Eigenvector candidates

Motivated by (4.84) we can take as a first ansatz

$$p^{n_0 n_1 n_2 n_3} = \delta_0^{n_1} \delta_0^{n_2} \delta^{n_0,n_3} c^{n_3}\,. \tag{4.88}$$

Inserting it into (4.78), we obtain

$$\begin{aligned}
\left(W_{inf}^2\right)_{n_0 n_1 n_2 n_3}^{m_0 m_1 m_2 m_3} \delta_0^{n_1} \delta_0^{n_2} \delta^{n_0,n_3} c^{n_3} = \\
&\Big(\delta^{m_0,m_3} \delta_0^{m_1} \delta_0^{m_2} (-2c^{m_3}(1+2m_3) + 2c^{m_3-1} m_3 + 2c^{m_3+1}(m_3+1)) \\
&+\delta^{m_0,m_3+2} \delta_0^{m_1} \delta_2^{m_2} \sqrt{2(m_3+1)(m_3+2)} (c^{m_3+2} + c^{m_3} - 2c^{m_3+1}) \\
&+\delta^{m_0,m_3+2} \delta_2^{m_1} \delta_0^{m_2} \sqrt{2(m_3+1)(m_3+2)} (c^{m_3+2} + c^{m_3} - 2c^{m_3+1})\Big) \\
&= -\,\mu^2 \delta_0^{m_1} \delta_0^{m_2} \delta^{m_0,m_3} c^{m_3}\,.
\end{aligned}$$

For the eigensystem to be valid we need to satisfy the two equations

$$2c^{m_3}(1+2m_3) - 2c^{m_3-1} m_3 - 2c^{m_3+1}(m_3+1) = \mu^2 c^{m_3} \tag{4.89}$$

$$c^{m_3+2} + c^{m_3} - 2c^{m_3+1} = 0 \tag{4.90}$$

which is solved only for $\mu^2 = 0$, and gives the solution for the coefficient

$$c^{m_3} = c^0 \tag{4.91}$$

A solution of (4.80) is then given by

$$p^{n_0 n_1 n_2 n_3} = c^0 \delta_0^{n_1} \delta_0^{n_2} \delta^{n_0,n_3} \tag{4.92}$$

which is simultaneously an eigenvector of J_3 with helicity $\lambda = 0$.

The result in (4.87) motivates another ansatz of the form

$$p^{m_0 m_1 m_2 m_3} = \sqrt{\frac{m_0!}{m_3!(m_0 - m_3)!}} \delta_{r-M}^{m_0 - m_3} C_{r,r}^{m_1 m_2} \, w^{m_3} \,. \tag{4.93}$$

A straightforward option is to choose $M = r$, and the proposed ansatz becomes

$$p^{n_0 n_1 n_2 n_3} = \delta^{n_0,n_3} C_{r,r}^{m_1 n_2} w^{n_3} \,. \tag{4.94}$$

Upon inserting into (4.80) we obtain two independent equations for w^{m_3}

$$w^{m_3}(1 + 2m_3) - w^{m_3 - 1} m_3 - w^{m_3 + 1}(m_3 + 1) = -\frac{\mu^2}{2(1+r)} w^{m_3} \tag{4.95}$$

$$w^{m_3 + 2} + w^{m_3} - 2w^{m_3 + 1} = 0 \tag{4.96}$$

The solution is given by $\mu^2 = 0$ and

$$w^{m_3} = w^0 \,. \tag{4.97}$$

This gives the polarization factor

$$p^{n_0 n_1 n_2 n_3} = \delta^{n_0,n_3} C_{r,r}^{n_1 n_2} \tag{4.98}$$

which is simultaneously an eigenvector of J_3 with helicity $\lambda = r$. The eigenvector (4.92) is thus a special case of (4.98). The norm of this solution is not finite and we can explicitly see that in the sum over Hermite functions in the auxiliary space, the eigenvector will contain a delta function. For instance, from (4.92) and the completeness identity of Hermite functions we find

$$\sum_{n_0,n_1,n_2,n_3=0}^{\infty} \delta^{n_0,n_3} \delta_0^{n_1} \delta_0^{n_2} f_{n_0}(u_0) f_{n_1}(u_1) f_{n_2}(u_2) f_{n_3}(u_3) = \delta^{(2)}(u_0 - u_3) e^{-\frac{u_1^2 + u_2^2}{2}} \,. \tag{4.99}$$

Through this approach we were able to obtain a solution to the equation (4.80) with a vanishing eigenvalue, and with an arbitrary integer helicity. This would correspond to an ordinary higher-spin massless field, but due to the infinite norm, the observables such as energy would not be finite. The non-finiteness of the norm could be a reflection of the fact that the eigenvalues μ^2 are continuous, and a different approach might be more suited for a complete characterization of the particle spectrum.

4.4 The quartic Casimir for an on-shell master field

Consider the linearized equations of motion one obtains if the integration over the auxiliary space is not performed prior to extremizing the action

$$\Box h_a(x,u) - \partial_a \partial^b h_b(x,u) = 0\,. \tag{4.100}$$

As in the previous section, we can fix the gauge to $\partial^a h_a(x,u) = 0$, and consider solutions representing plane waves directed along the z axis

$$h_a(x,u) = \epsilon_a \Phi(u) e^{ikx}\,, \tag{4.101}$$

where $k^a = (\omega, 0, 0, \omega)$ and $\epsilon_a = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$. Now, let's consider an active Lorentz transformation following the transformation properties (2.47)

$$h'_a(x^c, u_d) = \Lambda_a{}^b h_b((\Lambda^{-1}x)^c, (u \cdot \Lambda)_d) \tag{4.102}$$

If we expand the Lorentz transformation up to the first order

$$\Lambda^a{}_b \approx \delta^a_b + i\psi K^a{}_b$$

with ψ the expansion parameter[3], then $(\Lambda^{-1})^a{}_b \approx \delta^a_b - i\psi K^a{}_b$ and since $(\Lambda^{-1})^a{}_b = \Lambda_b{}^a$ it is true that $K^a{}_b = -K_b{}^a$. Through a simple expansion we get

$$h'_a(x^c, u_d) \approx h_a(x,u) + i\psi(K_a{}^b h_b(x,u) + K_c{}^b x^c \partial^x_b h_a(x,u) + K^b{}_c u_b \partial^c_u h_a(x,u))\,. \tag{4.103}$$

In case of our solution (4.101), the action of a generator of the Lorentz group, where $D(K)$ is a representation of the generator K, becomes

$$D(K) \cdot h_a(x,u) = \left(K_a{}^b \epsilon_a \Phi(u) + K_c{}^b x^c k_b \Phi(u) + K^b{}_c u_a \partial^c_u \Phi(u) \epsilon_b\right) e^{ikx}\,. \tag{4.104}$$

We would now like to examine the behavior of (4.101) under the action of the generators A, B of the little group $\mathfrak{iso}(2)$ with the reference momentum $k^a = (\omega, 0, 0, \omega)$. If we are able to find eigenfunctions of the mentioned generators, they will be the basis for the representation of the little group. Since A, B commute, the choice of finding their eigenfunctions is analogous to the plane-wave basis demonstrated in appendix A.2.

[3]The parameter can be an angle if Λ is a rotation, or rapidity in case of boosts.

It is straightforward to find the explicit vector representations for the operators $A = J_1 - K_2$ and $B = J_2 + K_1$.

$$A = \omega \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad B = \omega \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{4.105}$$

We see from (4.104) the three possible terms, of which only one will be non-trivial. Since $A^a{}_b k^b = B^a{}_b k^b = 0$, and $A^a{}_b \epsilon^b \propto k^b$, $B^a{}_b \epsilon^b \propto k^b$ which is a pure gauge contribution, the only term remaining in the equation (4.104) in case of the generators A and B is the last one, of the form $K^b{}_c u_b \partial_u^c \Phi(u)$. We now explicitly state the differential equations for A and B.

$$A \cdot \Phi(u) = u_a A^a{}_b \partial_u^b \Phi(u) \tag{4.106}$$
$$= i\omega(u_t - u_z)\frac{\partial}{\partial u_y} + i\omega u_y (\frac{\partial}{\partial u_t} + \frac{\partial}{\partial u_z})\Phi(u) \,. \tag{4.107}$$

In null-coordinates $u_+ = u_t + u_z, u_- = u_t - u_z$ it is simplified to

$$A \cdot \Phi(u) = i\omega \left[u_- \frac{\partial}{\partial u_y} + 2u_y \frac{\partial}{\partial u_+} \right] \Phi(u_+, u_-, u_x, u_y) \,. \tag{4.108}$$

The equation for B is similarly

$$B \cdot \Phi(u) = -i\omega \left[u_- \frac{\partial}{\partial u_x} + 2u_x \frac{\partial}{\partial u_+} \right] \Phi(u_+, u_-, u_x, u_y) \,. \tag{4.109}$$

Following (4.68) and (A.44)-(A.45), we want to find functions $\Phi(u)$ that satisfy the eigensystem

$$A \cdot \Phi(u) = a\, \Phi(u) \tag{4.110}$$
$$B \cdot \Phi(u) = b\, \Phi(u) \,. \tag{4.111}$$

The solutions to these equations for A and B separately are

$$\Phi_A(u) = \exp\left(\frac{-iau_y}{\omega u_-}\right) G_1 \left(u_-, u_x, \frac{-(u_t)^2 + (u_y)^2 + (u_z)^2}{2} \right) \tag{4.112}$$
$$\Phi_B(u) = \exp\left(\frac{ibu_x}{\omega u_-}\right) G_2 \left(u_-, u_y, \frac{-(u_t)^2 + (u_x)^2 + (u_z)^2}{2} \right) , \tag{4.113}$$

where G_1 and G_2 are arbitrary functions of their respective variables. We can write down a simultaneous solution with G an arbitrary function as

$$\Phi_{abr}(u) = \exp\left(i\frac{bu_x - au_y}{\omega u_-} \right) G_r \left(u_-, u_\mu u^\mu \right) \tag{4.114}$$

where a, b stand for the eigenvalues of A and B, and r stands for any additional indices that may be used to discriminate between different solutions. The explicit representation for W^2 is

$$W^2 = A^2 + B^2 \tag{4.115}$$

$$= -\,\omega^2 \left(u_-^2 \left(\frac{\partial^2}{\partial u_x^2} + \frac{\partial^2}{\partial u_y^2} \right) + 4u_- \left(u_x \frac{\partial^2}{\partial u_x} + u_y \frac{\partial}{\partial u_y} + 1 \right) \frac{\partial}{\partial u_+} + 4(u_x^2 + u_y^2) \frac{\partial^2}{\partial u_+^2} \right) \tag{4.116}$$

and we can immediately see that solutions (4.114) are eigenfunctions of the Casimir operator

$$W^2 \cdot \Phi(u) = (a^2 + b^2)\Phi(u) = \mu^2 \Phi(u)\,. \tag{4.117}$$

As expected from the properties of the little group, the eigenvalues of the Casimir W^2 are non-negative. Similar solutions were obtained in [74], in examining a scalar master field as a wavefunction of the continuous spin particle.

A complete orthonormal basis in the auxiliary space can be built from functions of the form (4.114), for a specific choice of the standard momentum. One possibility is to define

$$f_{abnl}(u) = \frac{1}{\sqrt{2\pi^2}} \exp\left(i \frac{b u_x - a u_y}{\omega u_-} \right) h_n(\omega u_-) h_l(\omega u_- u^2) \tag{4.118}$$

where $h_n(x)$ are any orthonormal and complete functions defined on $\mathbb{R}$, such as Hermite functions.

We prove that the functions $f_{abnl}(u)$ are orthonormal:

$$\int d^4u\, f_{a'b'n'l'}(u)^*\, f_{abnl}(u) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du_+\, h_{n'}(\omega u_+)^*\, h_n(\omega u_+) \tag{4.119}$$

$$\times \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2\, e^{-i\frac{(a-a')u_2 - (b-b')u_1}{\omega u_+}}$$

$$\times \int_{-\infty}^{\infty} du_-\, h_{l'}(\omega\, u_+ u^2)^*\, h_l(\omega u_+ u^2) \tag{4.120}$$

We can use a substitution

$$w \equiv \omega\, u_+ u^2 = \omega\, u_+ (u_+ u_- - u_1^2 - u_2^2) \tag{4.121}$$

which respects $w(u_- = \pm\infty) = \pm\infty$ to write the first integral

$$\int_{-\infty}^{\infty} du_-\, h_{l'}(\omega\, u_+ u^2)^*\, h_l(\omega\, u_+ u^2) = \frac{1}{\omega (u_+)^2} \int_{-\infty}^{\infty} dw\, h_{l'}(w)^*\, h_l(w) = \frac{\delta_{l'l}}{\omega (u_+)^2}\,. \tag{4.122}$$

The second integral gives

$$\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2\, e^{-i\frac{(a-a')u_2-(b-b')u_1}{\omega u_+}} = (2\pi\omega u_+)^2\, \delta(a'-a)\,\delta(b'-b) \tag{4.123}$$

and the third integral we have

$$\int_{-\infty}^{\infty} du_+\, h_{n'}(\omega u_+)^*\, h_n(\omega u_+) = \frac{1}{\omega^2}\delta_{nn'}\,. \tag{4.124}$$

Finally, we confirm that the basis functions are orthonormal

$$\int d^4u\, f_{a'b'n'l'}(u)^*\, f_{abnl}(u) = \delta(a'-a)\,\delta(b'-b)\,\delta_{l'l}\,\delta_{n'n} \tag{4.125}$$

We can also prove that the choice (4.118) is complete.

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db\, f_{abnl}(u')^*\, f_{abnl}(u) = \frac{1}{2\pi^2}\sum_l h_l(\omega u'_+ u'^2)^*\, h_l(\omega u_+ u^2)\\ \times\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db\, e^{i\frac{a(u'_2-u_2)-b(u'_1-u_1)}{\omega u_+}}\sum_n h_n(\omega u'_+)^*\, h_n(\omega u_+)\end{aligned} \tag{4.126}$$

Elementary functions such as Hermite satisfy completeness relations

$$\sum_{n=0}^{\infty} h_n(\omega u'_+)^*\, h_n(\omega u_+) = \delta(\omega(u'_+ - u_+)) = \frac{1}{|\omega|}\,\delta(u_+ - u'_+) \tag{4.127}$$

$$\sum_{l=0}^{\infty} h_l(\omega u'_+ u'^2)^*\, h_l(\omega u_+ u^2) = \delta(\omega u_+ u^2 - \omega u'_+ u'^2)\,. \tag{4.128}$$

With the exponential functions we have

$$\int_{-\infty}^{\infty} db\, e^{i\frac{b(u_1-u'_1)}{\omega u_+}} = (2\pi|\omega u_+|)\delta(u_1-u'_1) \tag{4.129}$$

$$\int_{-\infty}^{\infty} db\, e^{i\frac{a(u'_2-u_2)}{\omega u_+}} = (2\pi|\omega u_+|)\delta(u_2-u'_2)\,. \tag{4.130}$$

We can insert the results into (4.126) and obtain

$$\sum_n\sum_l\int da\int db\, f_{abnl}(u')^*\, f_{abnl}(u) = \tag{4.131}$$

$$\begin{aligned}&=2\omega(u_+)^2\,\delta(u_+-u'_+)\,\delta(u_1-u'_1)\,\delta(u_2-u'_2)\,\delta(\omega u_+u^2-\omega u'_+u'^2)\\ &=2\,\delta(u_+-u'_+)\,\delta(u_1-u'_1)\,\delta(u_2-u'_2)\,\omega(u_+)^2\,\delta(\omega(u_+)^2(u_--u'_-))\\ &=2\,\delta(u_+-u'_+)\,\delta(u_1-u'_1)\,\delta(u_2-u'_2)\,\delta(u_--u'_-)\\ &=\delta^4(u-u')\,,\end{aligned} \tag{4.132}$$

which is the completeness relation.

We conclude that the MHSYM theory supports a description of infinite-spin particles. On shell, a classical solution corresponding to a non-vanishing value of the quartic Casimir $W^2 = a^2 + b^2$ can be chosen as e.g.

$$h_a(x, u) = \epsilon_a f_{abnl}(k, u) e^{ikx} \tag{4.133}$$

where we have emphasized that the polarization functions have an implicit dependence on the momentum k^a.

Off-shell, the value of W^2 is not constrained in the present form of the MHS theory, instead, it seems that the theory can support a continuous range of parameters μ^2. Additionally, the basis functions (4.118) could point to a degeneracy in that different values of indices n, l in the choice (4.118) lead to the same eigenvalue of μ^2. Alternatively, they might correspond to some additional quantum numbers which are left to be uncovered. We leave as an open issue how to extend the definition of basis functions to general momenta and the possibility of constraining the theory to a single choice of μ^2.

Chapter 5

MHS gauge field models and conservation laws

5.1 Model building in the MHS gauge sector

Having developed the MHS formalism and having it cast into a more general form, we turn back to the question of constructing candidates for the theories based on the MHS symmetry. We use the covariant formulation introduced in chapter 2 and show that apart from the pure Yang Mills case, which will in the general formulation display different phases, we can include additional terms in the action which can provide new interactions, kinetic terms and solutions. We examine the conservation laws and identify an infinite tower of conserved charges. Finally, we display a connection of the MHSYM theory to matrix models.

5.1.1 General considerations

In the MHS gauge sector we take the degrees of freedom to be described by the MHS vielbein $e_a(x, u)$, which is an MHS tensor (transforms in the adjoint representation). The matter sector can be spanned by a set of matter fields collectively denoted by ψ which can be in different representations of the MHS symmetry (discussed in more detail in section 2). Correspondingly, the action is a sum of two parts,

$$S[e, \psi] = S_{\mathrm{hs}}[e] + S_{\mathrm{m}}[\psi, e] \tag{5.1}$$

one describing the pure MHS gauge theory, and the other the matter sector and its coupling to the MHS vielbein. We assume that the action is weakly non-local in the master space, by which we mean that both parts in (5.1) can be written in terms of a master space Lagrangian

$$S_{\mathrm{hs,m}} = \int d^d x \, d^d u \, L_{\mathrm{hs,m}}(x,u) \; . \tag{5.2}$$

We will consider here the simplest possibilities, which are (Moyal) polynomials manifestly covariant under the MHS symmetry. This means that $L_{\mathrm{hs}}(x,u)$ is an MHS tensor, while $L_{\mathrm{m}}(x,u)$ may be an MHS scalar or an MHS tensor, depending on the matter content. In both cases the MHS invariance of the action

$$\delta_\varepsilon S[e] = 0 \tag{5.3}$$

is manifestly guaranteed. In case of a master space Lagrangian being an MHS tensor, invariance of the action is a consequence of the trace property of the Moyal product (A.17).[1] In addition, we assume that the Lagrangian is a Lorentz scalar.

From (5.1) it follows that the EoM in the MHS gauge sector are

$$0 = \mathcal{F}^a_{\mathrm{hs}}(x,u) + \mathcal{J}^a_{\mathrm{m}}(x,u) \tag{5.4}$$

where

$$\mathcal{F}^a_{\mathrm{hs}}(x,u) = \frac{\delta S_{\mathrm{hs}}[e]}{\delta e_a(x,u)} \qquad , \qquad \mathcal{J}^a_{\mathrm{m}}(x,u) = \frac{\delta S_{\mathrm{m}}[\psi, e]}{\delta e_a(x,u)} \tag{5.5}$$

To obtain the EoM, the trace property of the Moyal product can be used, since variations of the fields must vanish at the boundary. This makes it equal whether the functional derivative is "left" or "right", as emphasized in (2.56).

The EoM in the matter sector are given by

$$0 = \frac{\delta S_{\mathrm{m}}[\psi, e]}{\delta \psi} \; . \tag{5.6}$$

Using (2.80), (2.48) and (A.17) the MHS variation of S_{hs} can be written as[2]

$$\begin{aligned} \delta_\varepsilon S_{\mathrm{hs}}[e] &= \int d^d x \, d^d u \, \mathcal{F}^a_{\mathrm{hs}}(x,u) \, \delta_\varepsilon e_a(x,u) \\ &= \int d^d x \, d^d u \, \mathcal{F}^a_{\mathrm{hs}}(x,u) \, \mathcal{D}^\star_a \varepsilon(x,u) \\ &= -\int d^d x \, d^d u \, \mathcal{D}^\star_a \mathcal{F}^a_{\mathrm{hs}}(x,u) \, \varepsilon(x,u) \; . \end{aligned} \tag{5.7}$$

[1] Assuming proper gauge transformations for which boundary terms vanish.

[2] It is assumed here that the MHS variation is a proper gauge transformation, in which case boundary terms in (2.48) and (A.17) vanish.

Then, from (5.3) we get the off-shell identity

$$\mathcal{D}_a^\star \mathcal{F}_{\rm hs}^a(x,u) = 0 \; . \tag{5.8}$$

Applying $\mathcal{D}_a^\star$ on EoM (5.4) and using (5.8) we get

$$\mathcal{D}_a^\star \mathcal{J}_{\rm m}^a(x,u) = 0 \tag{5.9}$$

which states that the matter master current is (on-shell) covariantly conserved.

5.1.2 MHSYM theory

Let us first analyze the MHS gauge sector. The matter sector is studied in chapter 6. The simplest acceptable Lagrangian term that is dynamical and has the flat vacuum $e_a(x,u) = u_a$ as a solution of the EoM it generates is

$$L_{\rm hs}(x,u) = \frac{1}{4g_{\rm ym}^2}\, T_{ab}(x,u) \star T^{ba}(x,u) \equiv L_{\rm ym}(x,u) \; . \tag{5.10}$$

This is the Lagrangian of the MHSYM theory already introduced in chapter 2. The corresponding action is

$$\begin{aligned} S_{\rm ym}[e] =& \frac{1}{4g_{\rm ym}^2} \int d^dx\, d^du\, T_{ab}(x,u) \star T^{ba}(x,u) \\ =& \frac{1}{4g_{\rm ym}^2} \int d^dx\, d^du\, [e_a \overset{\star}{,} e_b] \star [e^b \overset{\star}{,} e^a] \; . \end{aligned} \tag{5.11}$$

Under a generic variation $\delta e_a(x,u)$ that vanishes on the boundary of the integration volume, the MHSYM action transforms as

$$\begin{aligned} \delta S_{\rm ym}[e] =& \frac{1}{2g_{\rm ym}^2} \int d^dx\, d^du\, \{\mathcal{D}_b^\star T^{ba}(x,u) \overset{\star}{,} \delta e_a(x,u)\} \\ =& \frac{1}{g_{\rm ym}^2} \int d^dx\, d^du\, \mathcal{D}_b^\star T^{ba}(x,u)\, \delta e_a(x,u) \end{aligned} \tag{5.12}$$

which means that its contribution to the EoM is

$$\begin{aligned} \mathcal{F}_{\rm ym}^a(x,u) =& \frac{1}{g_{\rm ym}^2}\, \mathcal{D}_b^\star\, T^{ba}(x,u) \\ =& \frac{1}{g_{\rm ym}^2}\, \left[e_b(x,u) \overset{\star}{,} [e^a(x,u) \overset{\star}{,} e^b(x,u)]\right] \; . \end{aligned} \tag{5.13}$$

The EoM of the pure MHSYM theory are then

$$\mathcal{D}_b^\star\, T^{ba}(x,u) = 0 \; . \tag{5.14}$$

It is important to observe that the MHSYM theory is classically a scale-free theory from the master space perspective. If one takes the MHS vielbein to be dimensionless, then the pure MHSYM coupling constant g_{ym} is also dimensionless. Moreover, as the coupling constant can be absorbed by rescaling the MHS vielbein, it is a theory without an intrinsic coupling constant. In chapter 2 we have seen that in the YM formulation the theory was not scale-free. From the general perspective we can now understand that the scale was introduced by a choice of the empty flat vacuum. In this normalization, it is given by

$$e_a(x,u)_{\text{vac}} = \ell_h u_a \ . \tag{5.15}$$

Note that the scale ℓ_h can be changed by "canonical" scale transformations

$$e'_a(x,u) = e_a(\lambda x, u/\lambda) \tag{5.16}$$

which form a subgroup of MHS transformations. The Minkowski vacuum spontaneously breaks a part of the MHS symmetry.

In the perspective where the MHS vielbein is the fundamental variable we can observe nonequivalent possibilities for vacua in the MHSYM theory. They are solutions of the EoM which satisfy the condition[3]

$$T_{ab}(x,u) = 0 \tag{5.17}$$

We have already mentioned that the flat configuration $e_a(x,u) = u_a$ is, at least from the classical viewpoint, a well-defined Lorentz-invariant vacuum. However, it is not the case that all configurations satisfying (5.17) are MHS gauge equivalent to the flat vacuum. An obvious example is an "empty" configuration $e_a(x,u) = 0$ which is a fixed point of MHS gauge transformations. To obtain some insight into the structure of vacua, let us examine the vacua that are of the form

$$e_a(x,u) = M_a{}^{\mu} u_{\mu} \tag{5.18}$$

where M are arbitrary constant real $d \times d$ matrices. MHS transformations preserving the shape of these configurations have the gauge parameter master field of the form

$$\mathcal{E}_{\Lambda}(x,u) = x^{\mu} \Lambda_{\mu}{}^{\nu} u_{\nu} \tag{5.19}$$

where Λ is again an arbitrary constant real $d \times d$ matrix. We see that

$$[\mathcal{E}_{\Lambda}(x,u) \stackrel{\star}{,} u_{\mu}] = i \Lambda_{\mu}{}^{\nu} u_{\nu} \tag{5.20}$$

[3]In section 5.2 we show that all conserved charges, including the energy-momentum tensor, vanish for configurations satisfying (5.17).

from which it follows that under large gauge transformations the MHS vacuum solution $e_a(x,u) = u_a$ becomes

$$
\begin{aligned}
e_a^\Lambda(x,u) \equiv& e_\star^{-i\mathcal{E}_\Lambda(x,u)} \star u_a \star e_\star^{i\mathcal{E}_\Lambda(x,u)} \\
=& \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[\mathcal{E}_\Lambda(x,u) \overset{\star}{,} \left[\mathcal{E}_\Lambda(x,u) \overset{\star}{,} \ldots \left[\mathcal{E}_\Lambda(x,u) \overset{\star}{,} u_a\right]\right] \ldots\right] \\
=& \delta_a^\mu \, (e^\Lambda)_\mu{}^\nu u_\nu
\end{aligned}
\tag{5.21}
$$

If M cannot be written as an exponential of some matrix, the corresponding configuration (5.18) is not MHS gauge equivalent to the vacuum (5.15).

Using (5.21) it is easy to show that a large MHS transformation of a vacuum solution (5.18), with a parameter in the form 5.19, produces the same type of vacuum with matrix M^Λ given by

$$
M^\Lambda = M e^\Lambda \ , \tag{5.22}
$$

where matrix multiplication is assumed. A corollary is that two vacua of the type (5.18), which are defined with matrices of different rank, are MHS gauge inequivalent.

This analysis sugests that the MHSYM theory contains different phases. The flat vacuum $e_a(x,u) = u_a$ describes an empty flat (Minkowski) background and defines a geometric phase in the sense of interpretations offered in chapter 2 and 7. When expanded around the flat vacuum solution as in (2.70) the linear part of the EoM is second-order in spacetime derivatives, and in this phase the theory has a perturbative regime (in the coupling constant). In contrast, the configuration $e_a(x,u) = 0$ does not have an emergent regular geometric description and defines a non-perturbative strongly-coupled unbroken phase (it is the only vacuum with a trivial orbit with respect to the MHS transformations).

5.1.3 Beyond MHSYM theory

We will analyze possible generalizations of the MHSYM theory, formed by additional terms we can add to the pure MHSYM action. There is just one lower-dimensional term allowed by the MHS symmetry for a generic number of spacetime dimensions d,

$$
L_1(x,u) = -\frac{\lambda_1}{2}\, g(x,u) = -\frac{\lambda_1}{2}\, e_a(x,u) \star e^a(x,u) \tag{5.23}
$$

which produces the following EoM contribution

$$
\mathcal{F}_1^a(x,u) = -\lambda_1\, e^a(x,u) \ . \tag{5.24}
$$

This term is not dynamical and has the appearance of a mass term, but in the geometric phase it behaves as a generalized cosmological constant term (see also the discussion and analysis in section 7.3.2). When added to the MHSYM action, the flat configuration $e_a(x,u) = \delta_a^\mu u_\mu$ is no longer solution of EoM, so this term changes the vacuum in the geometric phase.

For general d there is one additional independent term, which is of the same dimension as $T_{ab}(x,u) \star T^{ab}(x,u)$,

$$L_2(x,u) = \frac{\lambda_2}{4}\, g(x,u) \star g(x,u) \tag{5.25}$$

where $g(x,u) = e_a(x,u) \star e^a(x,u)$ was defined in section 2.3.4, which would contribute the following EoM term

$$\mathcal{F}_2^a(x,u) = \frac{\lambda_2}{2}\, \{g(x,u) \stackrel{\star}{,} e^a(x,u)\}\ . \tag{5.26}$$

Similar to the generalized cosmological constant term, the effect of this addition removes the flat configuration $e_a(x,u) = u_a$ from the solution space. However, this term is also dynamical and so it is interesting to see how it contributes to the linearized EoM in the geometric phase. If we write

$$e_a(x,u) = e_a^{(0)}(x,u) + h_a(x,u) \tag{5.27}$$

where $e_a^{(0)}(x,u)$ is the solution of the EoM (a background), it is straightforward to show

$$\mathcal{F}_2^a(x,u) = \frac{\lambda_2}{2} \Big(\{g^{(0)}(x,u) \stackrel{\star}{,} h^a(x,u)\} + \big\{ \{e_b^{(0)}(x,u) \stackrel{\star}{,} h^b(x,u)\} \stackrel{\star}{,} e^a_{(0)}(x,u) \big\} + \mathcal{O}(h^2) \Big)\ .$$

In case of the simplest type of background belonging to the geometric phase,

$$e_a^{(0)}(x,u) = e_a^{(0)\mu}(x)\, u_\mu$$

the contribution to the linearized EoM is at most second-order in spacetime derivatives. If the background is not of this type, it must have an infinite Taylor expansion in u and as a consequence its contribution to the linearized EoM have an infinite number of terms with no bounds on order in spacetime derivatives.

Similarly, we can construct potential Lagrangian terms by taking higher polynomials in the MHS vielbein, all of them having a higher dimension in the geometric phase in $d > 4$ than the terms already discussed. There are two interesting terms of a topological origin. If the number of spacetime dimensions is even, $d = 2r$, there exists a Lorentz-scalar MHS tensor

$$P_r(x,u) = \epsilon^{a_1 b_1 \ldots a_r b_r}\, T_{a_1 b_1}(x,u) \star \cdots \star T_{a_r b_r}(x,u)$$

where $\epsilon^{a_1 \dots a_d}$ is the Levi-Civita symbol. As it is a generalization of the Chern term we refer to it as the MHS Chern tensor. It can be used to construct Lagrangian terms, the simplest being

$$L_P(x,u) = \lambda_P \, P_r(x,u) \; . \tag{5.28}$$

Note that this term is parity-odd. In $d = 4$ it has the same dimension as the MHSYM term. It is not hard to show that it is a topological term,

$$P_r(x,u) = D^\star_{a_1} \left(\epsilon^{a_1 b_1 \dots a_r b_r} \, e_{b_1}(x,u) \star T_{a_2 b_2}(x,u) \star \cdots \star T_{a_r b_r}(x,u) \right) \tag{5.29}$$

where we used (2.81) and (2.87). As a consequence it does not contribute to the (bulk) EoM, but may possibly lead to non-perturbative effects if the theory contains topologically non-trivial configurations analogous to instantons.

If the number of spacetime dimensions is odd, $d = 2r + 1$, we can construct the following Lagrangian term

$$L_{\mathrm{CS}}(x,u) = \epsilon^{a b_1 c_1 \dots b_r c_r} \left\{ e_a(x,u) \stackrel{\star}{,} T_{b_1 c_1}(x,u) \star \cdots \star T_{b_r c_r}(x,u) \right\} \; . \tag{5.30}$$

This tensor is parity-odd as well. From (5.29) it follows that it can be obtained as a boundary term from the MHS Chern term. It is thus natural to call it the MHS Chern-Simons tensor. It produces the following contribution to the EoM

$$\mathcal{F}^a_{\mathrm{CS}}(x,u) = d \, \epsilon^{a b_1 c_1 \dots b_r c_r} \, T_{b_1 c_1}(x,u) \star \cdots \star T_{b_n c_n}(x,u) \tag{5.31}$$

which we call the MHS Cotton tensor. In $d = 3$ the MHS Chern-Simons term has a lower dimension than the MHSYM term so it dominates in the IR regime. The EoM of the pure MHS Chern-Simons theory in $d = 3$ is

$$\epsilon^{abc} T_{bc}(x,u) = 0 \qquad \Rightarrow \qquad T_{bc}(x,u) = 0, \tag{5.32}$$

which shows that the MHSCS theory is topological.

5.2 Conservation laws and conserved charges in MHSYM theory

5.2.1 Covariant vs. non-covariant conservation laws

Here we would like to examine in more detail the question of conservation laws and conserved charges in MHS theories, taking MHSYM theory as an example. As is well-known, in a Lorentz covariant theory a current satisfying the continuity equation on-shell (i.e., with EoM applied)[4]

$$\partial_\mu J^\mu(x) \doteq 0 \tag{5.33}$$

encodes a local conservation of charge defined by

$$Q_V(t) = \int_V d^{d-1}\mathbf{x}\, J^0(x) \ . \tag{5.34}$$

We refer to (5.33) as the conservation law. There are situations, especially when local symmetries are present, in which the total charge identically vanishes for all physical configurations. In the case of local symmetries such trivial charges are connected to the proper gauge transformations. Gauge transformations usually contain a subgroup of improper gauge transformations that are connected to non-trivial conserved charges. As the conserved charges in gauge theories can be written as asymptotic integrals, improper gauge symmetries are recognized by their "soft" fall-off in the limit $r \to \infty$. Here we are interested in extracting conservation laws and non-trivial charges in the framework of the MHS symmetry.

In theories with local symmetries covariant conservation laws appear naturally. In the case of MHS symmetry they are defined in the master space and are of the form

$$\mathcal{D}^\star_a \mathcal{J}^a(x,u) \doteq 0 \tag{5.35}$$

where $\mathcal{D}^\star_a$ is the MHS covariant derivative and $\mathcal{J}^a(x,u)$ is an MHS (tensor) current. An example is the matter current $\mathcal{J}_{\mathrm{m}}(x,u)$, defined in (5.5), which is the source in the MHS vielbein EoM. In ordinary theories with non-commutative local symmetries, such as YM theory and GR, a covariant conservation law does not directly imply conserved charges. For instance, the matter energy-momentum tensor in GR is covariantly conserved, but this does not imply conservation of the matter energy and momentum. It is possible to construct a corresponding energy-momentum pseudo-tensor which is conserved in the sense of (5.33), but which also contains a contribution from the spin-2 sector. In the case of the MHS symmetry, the covariant conservation (5.35) automatically generates the conservation law (5.33). This is because the MHS covariant derivative is by the definition a Moyal commutator and every Moyal commutator is a total divergence in the master

[4]For equalities valid on-shell we use the symbol "$\doteq$". The equalities valid for generic field configurations satisfying proper boundary conditions are denoted by the simple equality sign "$=$".

space

$$0 \doteq \mathcal{D}^\star_a \mathcal{J}^a(x,u) = \partial^x_\mu A^\mu_{\mathcal{J}}(x,u) + \partial^\mu_u B^{\mathcal{J}}_\mu(x,u) \ . \tag{5.36}$$

Integrating both sides of (5.36) over the auxiliary space and assuming that boundary terms in the auxiliary space are zero, we conclude that

$$J^\mu(x) \equiv \int d^d u \, A^\mu_{\mathcal{J}}(x,u) \tag{5.37}$$

is conserved. While covariantly conserved master field currents can usually be written in closed and compact expressions, we see from (5.36) and the structure of the Moyal product that physically conserved spacetime currents $J^\mu(x)$ may have a rather involved and cumbersome form when written explicitly.

An explicit example of a conserved charge for matter fields in the MHS theory that can be obtained in this way is the one generated by a constant improper MHS transformation $\varepsilon(x,u) = \varepsilon = \text{const}$, which corresponds to the $U(1)$ subgroup of the MHS symmetry. The MHS vielbein is neutral (invariant) under its action and so does not contribute to the $U(1)$ charge. It is the only conserved charge with such properties that is generated by the MHS symmetry. We now pass to a detailed study of conservation laws in the case of the MHSYM theory.

5.2.2 Conserved currents from EoM

In the geometric phase, conserved charges are directly obtained from the EoM following the standard procedure used in ordinary YM theory and GR. Let us demonstrate this in the case of MHSYM theory coupled to matter whose EoM is

$$\frac{1}{g^2_{\text{ym}}} \mathcal{D}^\star_b F^{ba}(x,u) = \mathcal{J}^a_{\text{m}}(x,u) \ . \tag{5.38}$$

We now use (2.70) and move all nonlinear terms in $h_a(x,u)$ to the right hand side, obtaining

$$\frac{1}{g^2_{\text{ym}}} \partial^x_b F^{ba}_{(1)}(x,u) = \tilde{\mathcal{J}}^a(x,u) \tag{5.39}$$

where

$$F^{ab}_{(1)}(x,u) = \partial^x_a h_b(x,u) - \partial^x_b h_a(x,u) \tag{5.40}$$

and

$$\tilde{\mathcal{J}}^a(x,u) = \mathcal{J}^a_{\rm m}(x,u) - \frac{i}{g_{\rm ym}^2}\left(2[h^b \overset{\star}{;} \partial^x_b h^a] - [h_b \overset{\star}{;} \partial^a_x h^b] + [\partial^x_b h^b \overset{\star}{;} h^a]\right) - \\ - \frac{1}{g_{\rm ym}^2}\left[h^b \overset{\star}{;} [h^a \overset{\star}{;} h_b]\right] . \tag{5.41}$$

Taking $a = 0$ in (5.39) we get

$$\tilde{\mathcal{J}}^0(x,u) \doteq \frac{1}{g_{\rm ym}^2}\partial^x_j F^{j0}_{(1)}(x,u) \tag{5.42}$$

which is the MHS Gauss's law, while taking the spacetime divergence of Eq. (5.39) yields the continuity equation

$$\partial^x_a \tilde{\mathcal{J}}^a(x,u) \doteq 0 \tag{5.43}$$

showing that the current $\tilde{\mathcal{J}}(x,u)$ is conserved in the master space (before integrating over auxiliary space). From Gauss's law (5.42) it follows that the corresponding locally conserved charge can be written as a surface space integral

$$\tilde{Q}_V(t,u) = \int_V d^{d-1}\mathbf{x}\, \tilde{\mathcal{J}}^0(x,u) \tag{5.44}$$

$$\doteq \int_V d^{d-1}\mathbf{x}\, \partial^x_j F^{j0}_{(1)}(x,u)$$

$$\doteq \oint_{S(V)} d^{d-2}a_j\, F^{j0}_{(1)}(x,u) \tag{5.45}$$

which is Gauss's law in the integral form. Equation (5.43) encodes a tower of conserved charges. To see this, we Taylor expand both sides in auxiliary coordinates around $u = 0$ to obtain an infinite set of conserved charges

$$\tilde{Q}^{\mu_1\cdots\mu_n} \doteq \oint d^{d-2}a_j\, F^{j0\mu_1\cdots\mu_n}_{(1)}(x) \tag{5.46}$$

where

$$F^{j0}_{(1)}(x,u) = \sum_{n=0}^{\infty} F^{j0\mu_1\cdots\mu_n}_{(1)}(x)\, u_{\mu_1}\cdots u_{\mu_n} . \tag{5.47}$$

5.2.3 Conservation laws from the Noether method

Let us now construct conservation laws by applying the Noether method. For simplicity, we restrict ourselves to the pure MHSYM theory. First, using the MHSYM EoM

$$\mathcal{D}^\star_a T^{ab}(x,u) = 0 \tag{5.48}$$

we conclude that a generic on-shell variation of the MHSYM master Lagrangian can be written as

$$\delta L_{\rm ym}(x,u) \doteq -\frac{1}{2g_{\rm ym}^2}\mathcal{D}^\star_a\{T^{ab}(x,u) \overset{\star}{,} \delta e_b(x,u)\}\ . \tag{5.49}$$

On the other hand, under an MHS variation

$$\delta_\varepsilon e_a(x,u) = \mathcal{D}^\star_a \varepsilon(x,u) \tag{5.50}$$

the Lagrangian transforms as an MHS tensor,

$$\delta_\varepsilon L_{\rm ym}(x,u) = i[L_{\rm ym}(x,u) \overset{\star}{,} \varepsilon(x,u)]\ . \tag{5.51}$$

We now use (5.49) and (5.51) to write

$$0 \doteq \frac{1}{2g_{\rm ym}^2}\left(\mathcal{D}^\star_a\{T^{ab}(x,u) \overset{\star}{,} \mathcal{D}^\star_b\varepsilon(x,u)\} - \frac{i}{2}\big[T_{ab}(x,u)\star T^{ab}(x,u) \overset{\star}{,} \varepsilon(x,u)\big]\right). \tag{5.52}$$

As both terms on the right hand side are Moyal commutators the equation has the form

$$0 \doteq \partial^x_\mu A^\mu_\varepsilon(x,u) + \partial^\mu_u B^\varepsilon_\mu(x,u)\ . \tag{5.53}$$

Again, integrating over the auxiliary space and assuming that all boundary terms vanish, we obtain a standard conservation law (in the form of the continuity equation)

$$\partial^x_\mu J^\mu_\varepsilon(x) \doteq 0 \qquad , \qquad J^\mu_\varepsilon(x) \equiv \int d^d u\, A^\mu_\varepsilon(x,u)\ . \tag{5.54}$$

The corresponding conserved charges

$$Q_\varepsilon = \int d^{d-1}\mathbf{x}\, J^0_\varepsilon(x) \tag{5.55}$$

are non-trivial only for a small class of MHS parameters corresponding to improper gauge transformations. It is expected that rigid variations $\varepsilon(x,u) = \varepsilon(u)$, which we can expand as

$$\varepsilon(u) = \sum_{n=0}^\infty \xi^{\mu_1\cdots\mu_n} u_{\mu_1}\ldots u_{\mu_n} \tag{5.56}$$

with $\xi^{\mu_1\cdots\mu_n}$ constant and completely symmetric, fall into this class. [5] We have already analyzed the $n=0$ case, which does not affect MHS vielbein and so (5.52) becomes trivial ($0 \doteq 0$). Let us now consider $n \geq 1$ cases, by first constructing covariantly conserved

[5]The transformations (5.56) are surely not a complete set of improper MHS transformations. As we know from Maxwell's theory and GR, constant transformations are not the only ones generating non-trivial charges. A complete analysis of asymptotic symmetries is left for future work.

currents. For this we first have to find the covariantized form of the rigid MHS variation (5.56). Motivated by Jackiw's covariantization trick [79] we see that the simplest way to do this is by replacing $u_a \to e_a(x,u)$

$$\varepsilon(x,u) = \sum_{n=0}^{\infty} \xi^{a_1\cdots a_n} e_{a_1}(x,u) \star \ldots \star e_{a_n}(x,u) \tag{5.57}$$

where $\xi^{a_1\cdots a_n}$ is a constant tensor with symmetries guaranteeing reality of the MHS parameter $\varepsilon(x,u)$.[6] Now we use this in (5.52) where we want to write the second term on the right hand side as a covariant divergence (the first term is already in this form). We do this by using the identity

$$[A_1 \star \ldots \star A_n \overset{\star}{,} X] = \sum_{j=1}^{n} [A_j \overset{\star}{,} A_{j+1} \star \ldots \star A_n \star X \star A_1 \star \ldots \star A_{j-1}] \tag{5.58}$$

valid for generic master space functions $A_j(x,u)$ and $X(x,u)$, to write

$$\begin{aligned} & i\left[e_{a_1} \star \ldots \star e_{a_n} \overset{\star}{,} T_{ab} \star T^{ab}\right] \\ & \qquad = \sum_{j=1}^{n} \mathcal{D}^{\star}_{a_j}\left(e_{a_{j+1}} \star \ldots \star e_{a_n} \star T_{ab} \star T^{ab} \star e_{a_1} \star \ldots \star e_{a_{j-1}}\right). \end{aligned} \tag{5.59}$$

Using this in (5.52) we obtain

$$\mathcal{D}^{\star}_a \mathscr{T}^a_{\xi}(x,u) \doteq 0 \tag{5.60}$$

with the covariantly conserved currents given by

$$\begin{aligned} \mathscr{T}^a_{\xi} = \quad & \xi^{b_1\cdots b_n}\Bigg(\{T^{ac} \overset{\star}{,} \mathcal{D}^{\star}_c\left(e_{b_1} \star \ldots \star e_{b_n}\right)\} + \\ & + \frac{1}{2}\sum_{j=1}^{n} \delta^a_{b_j} e_{b_{j+1}} \star \ldots \star e_{b_n} \star T_{cd} \star T^{cd} \star e_{b_1} \star \ldots \star e_{b_{j-1}}\Bigg). \end{aligned} \tag{5.61}$$

The currents related to totally symmetric $\xi^{b_1\cdots b_n}$

$$\begin{aligned} \mathscr{T}^a{}_{b_1\cdots b_n} = \quad & \{T^{ac} \overset{\star}{,} \mathcal{D}^{\star}_c\left(e_{(b_1} \star \ldots \star e_{b_n)}\right)\} + \\ & + \frac{1}{2}\sum_{j=1}^{n} \delta^a_{(b_j} e_{b_{j+1}} \star \ldots \star e_{b_n} \star T_{|cd|} \star T^{cd} \star e_{b_1} \star \ldots \star e_{b_{j-1})} \end{aligned} \tag{5.62}$$

play a special role as they are obtained by covariantizing the rigid MHS symmetries. Another reason for its special status is that they have the softest behavior at spatial

[6] For $\varepsilon(x,u)$ in (5.57) to be real it has to be expressible purely in terms of Moyal commutators and/or anticommutators. If the number of anticommutators is odd, the parameter $\xi^{a_1\cdots a_n}$ is imaginary. When $\xi^{a_1\cdots a_n}$ is completely symmetric the expression (5.57) is a covariantization of (5.56).

infinity ($r \to \infty$) in the geometric phase, which means that they are main candidates for producing non-trivial charges. Since the corresponding conserved charges are described by totally symmetric tensors, they should be related to the charges (5.46) obtained by the previous method.

The $n = 1$ case in (5.56) corresponds to spacetime translations, leading to energy-momentum conservation, and is therefore of special importance. Fixing $n = 1$ in (5.62) we get the covariant master energy-momentum tensor

$$\mathscr{T}_b^{\,a}(x,u) = \{T^{ac} \stackrel{\star}{,} T_{bc}\} - \frac{1}{2}\,\eta_b^a\, T_{cd} \star T^{cd} \tag{5.63}$$

which is symmetric, and in $d = 4$ traceless. As expected, the obtained expression has the same form as in non-commutative field theories [80, 81, 82, 83].

5.3 MHSYM as a matrix theory

There is another way to represent MHS theories discussed above, which uses the connection between the Moyal product and the Weyl-ordered operator product well known from the phase space formulation of a quantized particle. If we define the Hilbert space $\mathcal{H}$ with the complete set of operators $\hat{x}^\mu, \hat{u}_\mu$ satisfying commutation relations

$$[\hat{x}^\mu, \hat{u}_\nu] = i\delta^\mu_\nu \quad , \quad [\hat{x}^\mu, \hat{x}^\nu] = 0 = [\hat{u}_\mu, \hat{u}_\nu] \tag{5.64}$$

there is a bijective map for a fixed ordering scheme (details in appendix A.2) between the set of linear operators on $\mathcal{H}$ and the set of functions on the master space, i.e.[7]

$$\mathrm{End}(\mathcal{H}) \ni \hat{O} \quad \longleftrightarrow \quad O(x,u) \in C^\infty(\mathcal{M} \times \mathcal{U}) \; . \tag{5.65}$$

If one defines a product of two operators with the (symmetric) Weyl ordering of x and u, its pull-back to the master space (through the map (5.65)) defines the Moyal product of corresponding master space functions (symbols)

$$\hat{O}_1\,\hat{O}_2 \quad \longleftrightarrow \quad O_1(x,u) \star O_2(x,u) \; . \tag{5.66}$$

This map is such that the trace of an operator is given by the integral of the corresponding function over the master space

$$\mathrm{tr}(\hat{O}) = \int d^dx\, \frac{d^du}{(2\pi)^d}\, O(x,u) \; . \tag{5.67}$$

[7] One usually refers to $O(x,u)$ as the symbol of operator $\hat{O}$.

Using this map it is now evident that all models for MHS theories can be written in this operator language, and therefore as a type of matrix models.[8] For example, the MHSYM theory can be written as

$$S_{\rm ym} = -\frac{(2\pi)^d}{4g_{\rm ym}}\,{\rm tr}\left([\hat{e}_a\,,\hat{e}_b][\hat{e}^a,\hat{e}^b]\right) \tag{5.68}$$

where $\hat{e}^a$ are operators on $\mathcal{H}$, components of a vector in the fundamental representation of $SO(1,d-1)$. The MHS symmetry is now represented by unitary linear operators

$$\hat{U}_{\mathcal{E}} = \exp(-i\hat{\mathcal{E}}) \tag{5.69}$$

which act on the MHS vielbein operator as

$$\hat{e}_a \to \hat{U}_{\mathcal{E}}\,\hat{e}_a\,\hat{U}_{\mathcal{E}}^\dagger \tag{5.70}$$

with all operator products defined with symmetric Weyl ordering.

[8]For an explicit proof that the Moyal product is a matrix product see [84].

Chapter 6

Coupling to matter and scattering

Within the MHS framework we can also seek to describe matter fields. Our original approach for gauging higher-spin symmetries described in section 2.1.2, started by an appropriate reformulation of the spacetime action for a massive scalar field in the master space language. It is surely then a valid approach to use such a formulation for matter, which we will call minimal matter. However, this is not the only possibility.

As we have seen in chapter 2, master space fields can transform in the adjoint or the fundamental representation of the MHS symmetry. For that reason, we can formulate models for matter as master fields transforming in the adjoint or fundamental representation, along with coupling them to the MHS field.

We will take the first steps in calculating scattering between matter fields mediated by the MHS field and find the results which depend on the type of matter.

6.1 Minimal matter

The first description of matter we wish to consider are known matter actions rewritten in the Moyal product language. As described in chapter 2, where (2.20) described a massive complex scalar field, we can see that using the master space formalism we can write minimally coupled matter actions in the form

$$S_m[\phi, e] = \int d^d x \, d^d u \, \mathrm{Tr}\big(W_\phi(x,u) \star K(e(x,u))\big) \tag{6.1}$$

where the trace is performed over Lorentz and internal indices carried by matter fields and the Wigner function can be written as

$$(W_\phi(x,u))_{rs} = \phi_r(x) \star \delta^d(u) \star \phi_s(x)^* \;, \tag{6.2}$$

with r, s standing for possible tensor/spinor or other internal indices. By construction, both the Wigner function and $K(e(x,u))$ are MHS tensors. It then follows that Lagrangians for minimally coupled matter, defined by (6.1), are also MHS tensors. Note that coupling to matter in this way explicitly breaks the translational symmetry in the auxiliary space (most easily noticed by the presence of $\delta^{(d)}(u)$ in the Wigner function). The matter action is also formally defined for non-geometric configurations.

To understand the nature of the minimal coupling in MHS theory in the geometric phase, we first use (2.70) to separate free and interacting parts of the action by writing

$$K(e(x,u)) = K(u) + K_{\text{int}}(h(x,u);u) \tag{6.3}$$

where the explicit dependence of K_{int} on u is present only for bosonic fields. Coupling to the MHS potential is linear for fermionic matter and quadratic for bosonic matter. Substituting this into (6.1) we get

$$S_m[\phi,h] = S_m^{(0)}[\phi] + S_m^{(\text{int})}[\phi,h] \tag{6.4}$$

where by definition $S_m^{(0)}$ is the action for the free field and the interaction term can be written as

$$S_m^{(\text{int})}[\phi,h] = \int d^dx\, d^du\, \big(\phi_r^*(x) \star K_{\text{int}}^{rs}(x,u) \star \phi_s(x)\big)\delta^d(u)\ . \tag{6.5}$$

Let us demonstrate the above construction on two important examples of matter; Dirac and Klein-Gordon fields. In case of a complex Klein-Gordon field we have already seen in (2.27) that in the geometric phase where $e_a(x,u) = u_a + h_a(x,u)$

$$K\big(e(x,u)\big) = e_a(x,u) \star e^a(x,u) - m^2 = u^2 - m^2 + 2\,u^a h_a(x,u) + h_a(x,u) \star h^a(x,u) \tag{6.6}$$

Then, following (6.3), we see that the interacting part is given by

$$K_{\text{int}}(x,u) = h(x,u) = 2\,u^a h_a(x,u) + h_a(x,u) \star h^a(x,u) \tag{6.7}$$

where $h(x,u)$ is a composite object obtained from the MHS potential, already introduced in (2.97). If we now use a Taylor expansion in the auxiliary coordinates,

$$h(x,u) = \sum_{s=0}^{\infty} h_{(s)}^{\mu_1\cdots\mu_s}(x)\, u_{\mu_1}\cdots u_{\mu_s} \tag{6.8}$$

we can find that the interacting part of the action is given by

$$S_m^{(\text{int})}[\varphi,h] = \sum_{s=0}^{\infty} \int d^dx\, J^{(s)}_{\mu_1\cdots\mu_s}(x)\, h_{(s)}^{\mu_1\cdots\mu_s}(x)\ . \tag{6.9}$$

where the spin-s currents are of the form

$$J^{(s)}_{\mu_1\cdots\mu_s}(x) = \frac{i^s}{2^s}\sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\partial^x_\mu\right)^k\varphi(x)\left(\partial^x_\mu\right)^{s-k}\varphi^*(x) \tag{6.10}$$

$$= \frac{i^s}{2^s}\,\varphi(x)^*\overset{\leftrightarrow}{\partial}_{\mu_1}\cdots\overset{\leftrightarrow}{\partial}_{\mu_s}\varphi(x)\ . \tag{6.11}$$

Details of this calculation can be found in appendix B.2. For concreteness, we report on the first few currents

$$J^{(0)}(x) = \varphi^*(x)\varphi(x) \tag{6.12}$$

$$J^{(1)}_\mu(x) = \frac{i}{2}(\partial_\mu\varphi(x)^*\varphi(x) - \varphi(x)^*\partial_\mu\varphi(x)) \tag{6.13}$$

$$J^{(2)}_{\mu_1\mu_2}(x) = \frac{-1}{4}(\varphi(x)\partial_{\mu_1}\partial_{\mu_2}\varphi(x)^* - 2\partial_{(\mu_1}\varphi(x)\partial_{\mu_2)}\varphi(x)^* + \partial_{\mu_1}\partial_{\mu_2}\varphi(x)\varphi(x)^*) \tag{6.14}$$

$$\vdots$$

This is the approach originally followed in [43, 44], where the linear coupling between a tower of higher-spin fields $h_{(s)}^{\mu_1\cdots\mu_s}(x)$ and the simple currents (6.10) was the starting point. The authors mention in [43] that their HS fields could be composite, and here through (6.7) we see that explicitly while arriving at the same simple currents through a Taylor expansion of the composite field $h(x,u)$.

In case of the Dirac field $\psi(x)$ we have

$$K_{\text{int}}(x,u) = -\gamma^0\gamma^a h_a(x,u)\ . \tag{6.15}$$

Taylor expanding the MHS potential $h_a(x,u)$ as in (2.70) and following the same steps for explicitly writing down the interaction part of the action one gets

$$S^{(\text{int})}_m[\psi,h] = \sum_{n=0}^{\infty}\int d^dx\, J^a_{(n)\mu_1\cdots\mu_n}(x)\, h_a^{(n)\mu_1\cdots\mu_n}(x) \tag{6.16}$$

where the HS currents ([3, 67]) are given by

$$J^a_{(n)\mu_1\cdots\mu_n}(x) = \frac{i^n}{2^n}\,\bar\psi(x)\gamma^a\overset{\leftrightarrow}{\partial}_{\mu_1}\cdots\overset{\leftrightarrow}{\partial}_{\mu_n}\psi(x)\ . \tag{6.17}$$

6.2 Master field matter

Another way to couple matter in an MHS symmetric way is to describe it by master fields, $\phi(x,u)$. This type of matter is necessary if one wants to introduce supersymmetry in the approach of [85]. The simplest representations are adjoint and fundamental.

6.2.1 Adjoint representation

Matter in the adjoint representation is described by MHS tensors, which means that the MHS covariant derivative is given by

$$\mathcal{D}^\star_a \phi(x,u) = i[e_a(x,u) \overset{\star}{,} \phi(x,u)] \ . \tag{6.18}$$

In case of minimal coupling the action is then constructed in the standard way, by substituting $\partial^x_a \to \mathcal{D}^\star_a$. This type of matter shares some properties with the MHS gauge sector action: the master fields are real, actions are also defined in the non-geometric phases, and they can be written in the form of matrix models.

Let us apply this to the free Majorana spin-1/2 field $\psi(x,u)$. The MHS action for minimal coupling is

$$S_M[\psi, e] = \frac{1}{2}\int d^dx\, d^du\, \bar{\psi}(x,u) \star (i\gamma^a \mathcal{D}^\star_a - M)\psi(x,u) \ . \tag{6.19}$$

In the operator formulation this is

$$S_M[\psi, e] = -\frac{(2\pi)^d}{2}\,\mathrm{tr}\left(\hat{\bar{\psi}}\big(\gamma^a[\hat{e}_a, \hat{\psi}] + M\hat{\psi}\big)\right) . \tag{6.20}$$

In case of a real scalar field the minimal coupling is described by the following action

$$S_s[\varphi, h] = \int d^dx\, d^du\, \left[\eta^{ab}(D^\star_a\varphi)^* \star D^\star_b\varphi - m^2\varphi^* \star \varphi - V_\star(\varphi^* \star \varphi)\right] . \tag{6.21}$$

6.2.2 Fundamental representation

Matter in the fundamental representation of MHS symmetry transforms as

$$\phi_{\mathcal{E}}(x,u) = e_\star^{-i\mathcal{E}(x,u)} \star \phi(x,u) \tag{6.22}$$

from which it follows

$$\phi_{\mathcal{E}}(x,u)^* = \phi(x,u)^* \star e_\star^{i\mathcal{E}(x,u)} \tag{6.23}$$

In the YM-like formalism the MHS covariant derivative in the fundamental representation is

$$D^\star_a\phi = \partial^x_a\phi + i\, h_a \star \phi \ . \tag{6.24}$$

It is simple to check the MHS covariance

$$(D^\star_a\phi)^{\mathcal{E}} = e_\star^{-i\mathcal{E}} \star D^\star_a\phi \ . \tag{6.25}$$

MHS invariants are constructed by Moyal-sandwiching MHS tensors between ϕ^* or $(D^\star_a\phi)^*$ from the left and ϕ or $D^\star_a\phi$ from the right. Using these invariants we can produce candidates for Lagrangian terms, with minimal coupling defined in the usual manner by a substitution of MHS covariant derivative for partial spacetime derivative in free field actions.

However, minimal prescription based on (6.24) can be defined only in the geometric phase. In addition, it is not natural from the the perspective of a matrix model formulation. These shortfalls can be avoided by using the prescription

$$\partial^x_a\phi(x,u) \to i\,e_a(x,u)\star\phi(x,u)\ . \tag{6.26}$$

From the relation

$$i\,e_a(x,u)\star\phi(x,u) = iu_a\phi(x,u) + \frac{1}{2}\partial^x_a\phi(x,u) + i\,h(x,u)\star\phi(x,u) \tag{6.27}$$

it is obvious that it differs from (6.24). To understand the origin of this degeneracy of minimal prescriptions, let us consider an example of the master Dirac field $\psi(x,u)$. In this case it is easy to show that

$$-\bar\psi\gamma^a\star e_a\star\psi = \frac{i}{2}\bar\psi\gamma^a\star D^\star_a\psi - \frac{i}{2}\overline{D^\star_a\psi}\star\gamma^a\psi + u_a\bar\psi\gamma^a\star\psi\ . \tag{6.28}$$

The first two terms on the right hand side produce the master Lagrangian kinetic term which one would obtain by the minimal coupling prescription based on (6.24), leading to the action

$$S_{D1}[\psi,e] = \int d^dx\,d^du\,\bar\psi(x,u)\star\big(i\gamma^aD^\star_a - M\big)\psi(x,u)\ . \tag{6.29}$$

On the left hand side of (6.28) is the expression which takes natural matrix model form when used in the action

$$S_{D2}[\hat\psi,\hat e] = -\mathrm{Tr}\left(\hat{\bar\psi}(\gamma^a\hat e_a + M)\hat\psi\right) \tag{6.30}$$

and is formally defined for all phases of the MHS theory (it also takes care of hermicity by automatism). The two actions differ already at the free field level, i.e., for $h_a(x,u)=0$. We now see that the difference between Lagrangians in two prescriptions is the third term on the right hand side of (6.28) which is an MHS scalar. Its existence is a consequence of the fact that Lagrangian terms for matter in the fundamental representation are MHS scalars, which means that they can be multiplied by functions of the auxiliary coordinates without breaking any of the important symmetries.[1]

Let us mention that mater fields in the fundamental representation have an additional peculiarity in that the rigid MHS variations with $n = 1$ ($s = 2$) act differently than in the case of the MHS vielbein and previously discussed realizations of matter,

$$\begin{aligned}\delta_{\varepsilon_{(1)}}\phi &= -\,i\,\varepsilon^{\mu}u_{\mu}\star\phi \\ &= -\,i\,\varepsilon^{\mu}u_{\mu}\,\phi - \frac{\varepsilon^{\mu}}{2}\,\partial^{x}_{\mu}\phi\,.\end{aligned} \tag{6.31}$$

We see that it does not describe spacetime translations. One consequence is that the MHS transformations in this case can be consistently truncated only to the lowest spin sector ($n = 0$) when master fields are Taylor-expanded around $u = 0$.

6.3 Tree-level scattering

The most accessible results we can obtain as potential observables are the scattering amplitudes in the lowest perturbation order. We will construct the Feynman rules for the MHS sector and the matter sector. We have already seen that the MHSYM action in the geometric phase, when linearized, becomes quite similar to the action of Maxwell's theory

$$S^{(2)}_{\rm ym} = -\,\frac{1}{4g^2_{\rm ym}}\int d^dx\,d^du\,F^{(2)ab}(x,u)F^{(2)}_{ab}(x,u)\,. \tag{6.32}$$

If we now pass to the dimensionless auxiliary coordinates, as introduced in (2.62)

$$\bar{u} = \ell_h u \qquad , \qquad \bar{g}_{\rm ym} = \ell_h^{d/2}\,g_{\rm ym} \qquad , \qquad \bar{h}_a = h_a/\bar{g}_{\rm ym} \tag{6.33}$$

and restrict our attention to $d = 4$ in which the coupling constant becomes dimensionless, we can rewrite the linearized action as

$$S^{(2)}_{\rm ym} = \frac{1}{2}\int d^4x\,d^4\bar{u}\,(\partial_a\bar{h}_b(x,\bar{u}) - \partial_b\bar{h}_a(x,\bar{u}))(\partial^a\bar{h}^b(x,\bar{u}) - \partial^b\bar{h}^a(x,\bar{u}))\,. \tag{6.34}$$

By using an orthonormal basis of functions in the auxiliary space $\{f_r(\bar{u})\}$,

$$\int d^d\bar{u}\,f_r(\bar{u})\,f_s(\bar{u}) = \delta_{rs} \tag{6.35}$$

[1]Of course we should be careful not to break symmetries which we would like to preserve, such as Lorentz symmetry and translations in spacetime. The symmetry under translations in the auxiliary space is broken by such multiplications, but we choose not to protect this symmetry.

to expand master fields as

$$\bar{h}_a(x,\bar{u}) = \sum_r \bar{h}^{(r)}_a(x)\, f_r(\bar{u}) \tag{6.36}$$

we can obtain a very simple expression for the linearized action

$$\begin{aligned} S^{(2)}_{\mathrm{ym}} =& \frac{1}{2}\int d^4x\, d^4\bar{u} \sum_{r,s} f_r(\bar{u}) f_s(\bar{u})(\partial_a \bar{h}^{(r)}_b(x,\bar{u}) - \partial_b \bar{h}^{(r)}_a(x,\bar{u}))(\partial^a \bar{h}^{(s)b}(x,\bar{u}) - \partial^b \bar{h}^{(s)a}(x,\bar{u})) \\ =& \frac{1}{2}\sum_{r,s}\delta_{rs}\int d^4x\, (\partial_a \bar{h}^{(r)}_b(x,\bar{u}) - \partial_b \bar{h}^{(r)}_a(x,\bar{u}))(\partial^a \bar{h}^{(s)b}(x,\bar{u}) - \partial^b \bar{h}^{(s)a}(x,\bar{u}))\,. \end{aligned} \tag{6.37}$$

Owing to the formal similarity to Maxwell's action, we find a simple expression for the propagator

$$D^{r,s}_{ab}(k) = D^{(\mathrm{QED})}_{ab}\delta_{r,s} \tag{6.38}$$

where $D^{(\mathrm{QED})}_{ab}$ is the usual propagator in quantum electrodynamics. For tree-level diagrams, this is the only Feynman rule we need regarding the gauge sector.

6.3.1 Minimal matter

We take as an example a single Dirac field $\psi(x)$ coupled to the MHS gauge field as in (6.1). By using the integral representation of the Moyal product (A.13) the interaction term becomes

$$\begin{aligned} S_{D,\mathrm{int}}[\psi,\bar{h}] =& -\bar{g}_{\mathrm{ym}}\sum_r \int d^dx \int d^d\bar{u}\, \bar{\psi}(x) \star \gamma^a \left[\bar{h}^{(r)}_a(x) f_r(\bar{u})\right] \star \psi(x)\, \delta^d(\bar{u}) \\ = -\bar{g}_{\mathrm{ym}}\sum_r \int & d^dx\, d^dy d^dz \frac{d^dw}{(2\pi)^d}\frac{d^dv}{(2\pi)^d} e^{i\frac{y}{\ell_h}\bar{w} - i\frac{z}{\ell_h}\bar{v}} \bar{\psi}(x+\frac{y}{2})\gamma^a h^r_a(x) f_r(\bar{v}+\bar{w})\psi(x+\frac{z}{2})\,. \end{aligned}$$

We now go to the momentum space with

$$\bar{\psi}(x+\frac{y}{2}) = \int \frac{d^dk}{(2\pi)^d} e^{ik(x+\frac{y}{2})}\bar{\Psi}(k) \tag{6.39}$$

$$\psi(x+\frac{z}{2}) = \int \frac{d^dk'}{(2\pi)^d} e^{ik'(x+\frac{z}{2})}\Psi(k') \tag{6.40}$$

$$h^{(r)}_a(x) = \int \frac{d^dq}{(2\pi)^d} e^{iqx} H^{(r)}_a(q) \tag{6.41}$$

and we can rewrite the interaction term as

$$S_{D,\mathrm{int}}[\psi,\bar{h}] = -\bar{g}_{\mathrm{ym}}\sum_r \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\delta^{(d)}(k+k'+q)\bar{\Psi}(k)\gamma^a H^{(r)}_a(q) f_r(\ell_h\frac{k'-k}{2})\Psi(k') \tag{6.42}$$

The easiest way to determine the Feynman rules for the vertices is to compare this expression to the interaction term in QED, rewritten in momentum space

$$\begin{aligned} S_{QED,int}[\psi, A] &= -\, e \int d^d x\, \bar{\psi(x)} \gamma^a A_a(x) \psi(x) \\ &= -\, e \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \delta^{(d)}(k + k' + q) \bar{\Psi}(k) \gamma^a A_a(q) \Psi(k') \end{aligned}$$

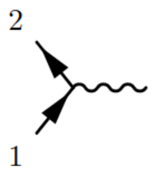
We now recognize that the vertex of the minimal coupling of simple matter to the MHS gauge field contains the same structure as in QED multiplied with a momentum dependent function,

$$V_r^a(k, k') = i\bar{g}_{\rm ym} \gamma^a f_r(\ell_h \frac{k' - k}{2})\,. \tag{6.43}$$

The basis functions (e.g. Hermite functions) vanish in the limit $|\bar{u}| \to \infty$ faster than any power, making the UV limit soft. The formula for the MHS vertex factor (6.43) suggests that the UV behavior of the MHSYM should be better than in QED.

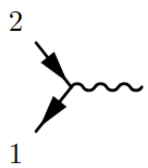
We wish to consider the simplest amplitude for an elastic scattering of fermions $f_1 f_2 \to f_3 f_4$, and for that purpose, there are two relevant Feynman rules, explicitly stated with their non-trivial momentum dependence (straight lines are fermion lines, while the wiggly lines correspond to the MHS field).

- Incoming particle with momentum p_1, outgoing particle with momentum p_2



$$= i\bar{g}_{\rm ym} \gamma^a f_r(-\ell_h \frac{p_1 + p_2}{2}) \tag{6.44}$$

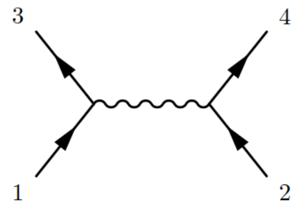
- Incoming antiparticle with momentum p_1, outgoing antiparticle with momentum p_2



$$= i\bar{g}_{\rm ym} \gamma^a f_r(\ell_h \frac{p_2 + p_1}{2}) \tag{6.45}$$

The fermion propagator as well as external lines are the same as in QED.

The t-channel amplitude corresponds to the following diagram



whose contribution is

$$
\begin{aligned}
i\mathcal{M}_t =&\sum_{r,s} u(p_1) i\bar{g}_{\rm ym}\gamma^a f_r(-\ell_h \frac{p_1+p_3}{2})\bar{u}(p_3)\frac{-i\eta_{ab}\delta_{rs}}{(p_1-p_3)^2}u(p_2)i\bar{g}_{\rm ym}\gamma^b f_s(-\ell_h\frac{p_2+p_4}{2})\\
=& i\mathcal{M}_t^{QED}\sum_{r,s}\delta_{rs}f_r(-\ell_h\frac{p_1+p_3}{2})f_s(-\ell_h\frac{p_2+p_4}{2})\\
=& i\mathcal{M}_t^{QED}\delta^{(d)}\left(\frac{\ell_h}{2}(p_1+p_3-p_2-p_4)\right)
\end{aligned}
\tag{6.46}
$$

where $\mathcal{M}_t^{QED}$ is of the form expected in electrodynamics and where we have employed the Feynman gauge and used the completeness relation

$$
\sum_r f_r(\bar{u})\, f_r(\bar{v}) = \delta^d(\bar{u}-\bar{v})\,. \tag{6.47}
$$

We can also use the momentum conservation condition

$$
p_1+p_2 = p_3+p_4 \tag{6.48}
$$

to finally express

$$
i\mathcal{M}_t = i\mathcal{M}_t^{QED}\delta^{(d)}(\ell_h(p_3-p_2))\,. \tag{6.49}
$$

The u-channel consists of replacing $3 \leftrightarrow 4$, which enables us to write down the full tree level amplitude

$$
i\mathcal{M} = i(\mathcal{M}_t - \mathcal{M}_u) \tag{6.50}
$$

$$
= i(\mathcal{M}_t^{QED}\delta^{(d)}(\ell_h(p_3-p_2)) - \mathcal{M}_u^{QED}\delta^{(d)}(\ell_h(p_4-p_2))) \tag{6.51}
$$

It is vanishing unless the set of momenta in the final state is the same as the set of momenta in the initial state. This result is expected from the viewpoint of the Coleman-Mandula theorem, despite the fact that MHS theory does not fulfill all assumptions of the theorem. The result is also interesting from the perspective of the search for the dark matter candidates in cosmology.[2]

The presence of the Dirac delta functions in the amplitude is a feature shared by recent approaches to constructing a higher spin theory in flat spacetime under the name of

[2]For recent speculations that higher-spin particles may describe dark matter see [86].

chiral higher spin gravity. In particular, in [87] they have identified appropriate algebraic structures for n-point amplitudes of higher spin fields in flat spacetime using the spinor-helicity formalism. It was argued that higher spin amplitudes should be distributions of momenta supported only on the vanishing values of the Mandelstam variables, and they support this claim by identifying inside the amplitudes a delta function dependence on the spinor-helicity variables related to field momentum. Similarly, they conclude that the amplitudes are non-vanishing only for collinear momenta.

Focusing our attention on (6.47), we could envisage a way out of the appearance of Dirac delta functions by an appropriate restriction of the possible MHS particles allowed in the interaction. So far, we have identified a single consistent reduction of the configuration space of the MHSYM theory and it comes from restricting to odd functions in the auxiliary space $e_a(x,-u) = -e_a(x,u)$ ("truncation to spin-even sector"). In this case the basis functions $f_r(\bar{u})$ are also odd, $f_r(-\bar{u}) = -f_r(\bar{u})$, and the completeness relation becomes

$$\sum_r f_r(\bar{u})\, f_r(\bar{v}) = \frac{1}{2}\left(\delta^d(\bar{u}-\bar{v}) - \delta^d(\bar{u}+\bar{v})\right). \tag{6.52}$$

The t-channel amplitude is now

$$\mathcal{M}_t = \frac{1}{2}\mathcal{M}_t^{(\mathrm{QED})}\left(\delta^d\big(\ell_h(p_1-p_3)\big) - \delta^d\big(\ell_h(p_1+p_2)\big)\right). \tag{6.53}$$

The main conclusion, that the amplitude is ultralocal in momentum space, remains the same.

6.3.2 Master space matter in the fundamental representation

We take that matter is represented by a single master Dirac field $\psi(x,\bar{u})$ in the fundamental representation of the MHS symmetry (see Sec. 6.2). The free action is simply

$$S_{D,0}[\psi] = \int d^dx\, d^d\bar{u}\, \bar{\psi}(x,\bar{u})\gamma^a\partial_a^x\psi(x,\bar{u}) \tag{6.54}$$

while the interaction term is

$$\begin{aligned} S_{\mathrm{D,int}}[\psi,\bar{h}] &= -\,\bar{g}_{\mathrm{ym}}\int d^dx\, d^d\bar{u}\, \bar{\psi}(x,\bar{u}) \star \left(\gamma^a\bar{h}_a(x,\bar{u})\right) \star \psi(x,\bar{u}) \\ &= -\,\bar{g}_{\mathrm{ym}}\int d^dx\, d^d\bar{u}\, \mathrm{Tr}\left(\psi(x,\bar{u}) \star \bar{\psi}(x,\bar{u})\gamma^a\right)\bar{h}_a(x,\bar{u}) \end{aligned} \tag{6.55}$$

where Tr denotes the trace over spinor indices. Now we also expand the master Dirac field in the orthonormal basis in the auxiliary space

$$\psi_\beta(x,\bar{u}) = \sum_j \psi_\beta^{(j)}(x)\, f_j(\bar{u}) \tag{6.56}$$

where β is a spinor index. The basis used for the matter field does not have to be the same as the one used for the MHS potential.

The interaction terms can give us the Feynman rules for the vertex in the same way, we evaluate the Moyal products and pass to the momentum space

$$S_{\mathrm{D,int}}[\psi,\bar{h}] = -\bar{g}_{\mathrm{ym}} \int d^d\bar{u} \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d} \tag{6.57}$$

$$\times \bar{\psi}^{(i)}(k)\gamma^a \bar{h}^{(r)}_a(q)\psi^{(j)}(k') f_i^*(\bar{u}-\ell_h\frac{k'}{2}) f_j(\bar{u}+\ell_h\frac{k}{2}) f_r(\bar{u}) \tag{6.58}$$

Again, we find a momentum-dependent vertex function

$$V^a_{ijr}(k,k') = -\bar{g}_{\mathrm{ym}}\gamma^a \int d^du\, f_i^*(\bar{u}-\ell_h\frac{k'}{2}) f_j(\bar{u}+\ell_h\frac{k}{2}) f_r(\bar{u}) \tag{6.59}$$

Following the same steps as above, we can approach to calculate an amplitude for a scattering of the form $ff \to ff$. The momentum prescriptions would also follow (6.44)-(6.45). For the t-channel diagram with momenta labeled as in (6.3.1), we obtain the amplitude

$$\mathcal{M}_t = \mathcal{M}_t^{(\mathrm{QED})} A_t^{(ij)} \tag{6.60}$$

where

$$\begin{aligned} A_t^{(ij)} &= \sum_r V^{(\mathrm{hs})}_{i_1 j_1 r}(p_3,p_1)\, V^{(\mathrm{hs})}_{i_2 j_2 r}(p_4,p_2) \\ &= \int d^d\bar{u}\, f_{i_1}^*(\bar{u}+\frac{\ell_h}{2}p_1) f_{j_1}(\bar{u}+\frac{\ell_h}{2}p_3) \int d^d\bar{v}\, f_{i_2}^*(\bar{v}+\frac{\ell_h}{2}p_2) f_{j_1}(\bar{v}+\frac{\ell_h}{2}p_4) \sum_r f_r(\bar{u}) f_r(\bar{v}) \\ &= \int d^d\bar{u}\, f_{i_1}^*(\bar{u}+\frac{\ell_h}{2}p_1)\, f_{j_1}(\bar{u}+\frac{\ell_h}{2}p_4)\, f_{i_2}^*(\bar{u}+\frac{\ell_h}{2}p_2)\, f_{j_2}(\bar{u}+\frac{\ell_h}{2}p_4)\,. \end{aligned} \tag{6.61}$$

In passing from second to third line the completeness relation was used. The u-channel contribution to the amplitude is obtained from (6.60) and (6.61) by exchanging $p'_1 \leftrightarrow p'_2$ and $i_1 \leftrightarrow i_2$. The total tree-level amplitude is

$$\mathcal{M}_{\mathrm{tree}} = \mathcal{M}_t - \mathcal{M}_u\,. \tag{6.62}$$

There are no Dirac-delta functions which are present in the case of simple spacetime matter. The integral in (6.61) is convergent and, due to the asymptotic fall-off of functions $f_j(\bar{u})$ when $|\bar{u}|\to\infty$, the MHS contribution certainly makes the UV behavior softer when compared to the standard spinor QED. The basis functions can be chosen in the form

$$f_r(\bar{u}) = P_r(\bar{u}) e^{-(\ell_h \bar{u})^2}\,, \tag{6.63}$$

where P_r are polynomials. Using this in (6.61) we can conclude that 4-point amplitudes will have the following form

$$A_t^{(ij)} = P^{(ij)}(\ell_h p) \exp\left(-\frac{\ell_h^2}{16}\sum_{i,j=1}^{4}(p_i - p_j)^2\right) , \tag{6.64}$$

where $P^{(ij)}$ are polynomial functions including a normalization factor from the $\bar{u}$ integration. The exponential factor makes the UV behavior much softer compared to QED.

Chapter 7

Low spin sector and induced geometry

To gain more insight into the structures inside the MHS theory, we now focus on the low spin sector; meaning the first two terms in a Taylor expansion of master fields in the auxiliary space. This truncation is consistent at the level of equations of motion since the MHS algebra is closed under variations with such truncated parameters;

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{i[\varepsilon_2 \overset{\star}{,} \varepsilon_1]}\,, \tag{7.1}$$

and if $\varepsilon_1(x,u) = \epsilon_1(x) + \varepsilon_1^\mu(x)\, u_\mu$ and $\varepsilon_2(x,u) = \epsilon_2(x) + \varepsilon_2^\mu(x)\, u_\mu$ we find

$$i[\varepsilon_2 \overset{\star}{,} \varepsilon_1] = (\varepsilon_1^\mu \partial_\mu \epsilon_2 - \varepsilon_2^\mu \partial_\mu \epsilon_1) + (\varepsilon_1^\nu \partial_\nu \varepsilon_2^\mu - \varepsilon_2^\nu \partial_\nu \varepsilon_1^\mu) u_\mu\,. \tag{7.2}$$

With an identification of geometric structures appearing in the low spin equations and the MHS covariant derivative, we arrive at an induced geometric picture and display the relation to teleparallel geometry. Finally, we find additional exact vacuum solutions to the equations of motion which are of the form motivated by the Minkowski vacuum.

7.1 Emergent geometry in the MHS theory

We have seen in chapter 6 that the minimal way to incorporate interacting matter inside the MHS framework leads to the picture in which matter perceives spacetime fields obtained by Taylor expanding the MHS vielbein in the auxiliary space

$$e_a(x,u) = \sum_{n=0}^{\infty} e_a^{(n)\mu_1\ldots\mu_n}(x)\, u_{\mu_1}\cdots u_{\mu_n} \tag{7.3}$$

as a HS background. From this viewpoint the lowest two components, $e_a^{(1)}(x)$ and $e_a^{(1)\mu}(x)$, play the roles of the $U(1)$ potential and the emergent spacetime vielbein, respectively. In this section we focus on the low-spin sector $n \leq 1$ ($s \leq 2$), with the goal of finding out what sort of spacetime geometry emerges from the MHS framework.

For this reason we write the expansion (7.3) in the form

$$e_a(x,u) = A_a(x) + E_a{}^{\mu}(x)\, u_\mu + \dots \tag{7.4}$$

and similarly for the MHS variation parameter

$$\varepsilon(x,u) = \epsilon(x) + \varepsilon^{\mu}(x)\, u_\mu + \dots \tag{7.5}$$

and in all expressions ignore higher spin components (with $n > 1$) denoted above by ellipses. We noted in chapter 4 that the truncation to the low-spin sector is apparently consistent at the level of MHS symmetry and EoM. However, we should keep in mind that such truncated configurations are not physical (see section 4.1.1), so our findings based on this truncation serve only as a small window to possible geometric underpinnings of the MHS theory.

If $B(x,u)$ and $C(x,u)$ are generic master fields, their Moyal bracket truncated to the low spin sector,

$$\begin{aligned} i[B(x,u) \stackrel{\star}{,} C(x,u)] =& \frac{\partial C(x,u)}{\partial x^\mu}\frac{\partial B(x,u)}{\partial u_\mu} - \frac{\partial B(x,u)}{\partial x^\mu}\frac{\partial C(x,u)}{\partial u_\mu} + \dots \\ =& -\{B(x,u), C(x,u)\}_{\mathrm{PB}} + \dots \end{aligned} \tag{7.6}$$

is given by the Poisson bracket, where the master space plays the role of the phase space. From (7.6) it follows that the set of spin-2 truncated master fields is closed under the Moyal bracket. The Taylor expansion of (7.6) is given by

$$\begin{aligned} i[B(x,u) \stackrel{\star}{,} C(x,u)] =& B^{\nu}_{(1)}(x)\, \partial_\nu C_{(0)}(x) - C^{\nu}_{(1)}(x)\, \partial_\nu B_{(0)}(x) \\ &+ \Big(B^{\nu}_{(1)}(x)\, \partial_\nu C^{\mu}_{(1)}(x) - C^{\nu}_{(1)}(x)\, \partial_\nu B^{\mu}_{(1)}(x) \Big)\, u_\mu + \dots \end{aligned} \tag{7.7}$$

$$= \pounds_{B_{(1)}} C_{(0)} - \pounds_{C_{(1)}} B_{(0)} + \left(\pounds_{B_{(1)}} C_{(1)}\right)^{\mu} u_\mu + \dots \tag{7.8}$$

The last line is obtained by recognizing the differential-geometric structure, with Lie derivatives treating $B_{(0)}(x)$ and $C_{(0)}(x)$ as scalar fields and $A^{\mu}_{(1)}(x)$ and $B^{\mu}_{(1)}(x)$ as vector fields on the spacetime manifold. We will see below that this is generally true in our construction – all expressions truncated to the low-spin sector ($s \leq 2$) are going to be diff-covariant.

Let us apply this to the MHS variation of the MHS vielbein (2.71). Using (7.4), (7.5) and (7.8) we obtain that the low-spin spacetime fields transform as

$$\delta_\epsilon A_a(x) = \pounds_{E_a}\epsilon(x) \tag{7.9}$$

$$\delta_\varepsilon A_a(x) = -\pounds_\varepsilon A_a(x) \tag{7.10}$$

$$\delta_\epsilon E_a{}^\mu(x) = 0 \tag{7.11}$$

$$\delta_\varepsilon E_a{}^\mu(x) = \left(\pounds_{E_a}\varepsilon\right)^\mu(x) \tag{7.12}$$

We now see that the MHS variation with $n = 1$ acts as an infinitesimal diffeomorphism defined by

$$x'^\mu = x^\mu + \varepsilon^\mu(x) \tag{7.13}$$

under which $E_a{}^\mu(x)$ behaves as a set of vector fields, while $A_a(x)$ behaves as a set of scalars. Assuming that the frame $E_a{}^\mu(x)$ is regular, i.e., there exists a co-frame $E^a{}_\mu(x)$ satisfying

$$E^a{}_\mu(x)E_a{}^\nu(x) = \delta^\nu_\mu \qquad , \qquad E^b{}_\mu(x)E_a{}^\mu(x) = \delta^b_a \tag{7.14}$$

an MHS variation with $n = 0$ acts on $A_\mu(x) = E^a{}_\mu(x)\, A_a(x)$ as

$$\delta_\epsilon A_\mu(x) = -\partial_\mu\epsilon(x) \tag{7.15}$$

while the frame $E_a{}^\mu(x)$ is invariant. Taken all together, $E_a{}^\mu(x)$ can be identified as the (inverse) vielbein, while $A_a(x)$ can be identified as a $U(1)$ gauge potential vector field in the non-coordinate basis of the vielbein. The $n = 0$ MHS variations are infinitesimal $U(1)$ gauge transformations, while $n = 1$ MHS variations are infinitesimal diffeomorphisms. Note that this interpretation is only valid in the geometric phase, in which the frame $E_a{}^\mu(x)$ invertible, to which we now turn our attention.

A word of caution is necessary here. If we keep higher spin contributions, the diff-covariant structure, at least as defined in the standard way, is apparently lost. This can be traced to the mixing (or twisting) of the HS transformations. These effects can be seen by analyzing $n = 1$ finite (large) MHS transformations of MHS tensors,

$$\mathcal{E}(x,u) = \mathcal{E}^\mu(x)\, u_\mu \tag{7.16}$$

where for the sake of simplicity we assume that the $n = 0$ component of the MHS tensor is vanishing

$$X_{a\ldots}(x,u) = X^{(1)\mu}_{a\ldots}(x)\, u_\mu + \mathcal{O}(u^2) \tag{7.17}$$

From the definition of large MHS transformations

$$X^{\mathcal{E}}_{ab\cdots}(x,u) = e_\star^{-i\,\mathcal{E}(x,u)} \star X_{ab\cdots}(x,u) \star e_\star^{i\,\mathcal{E}(x,u)} \tag{7.18}$$

and the Baker-Campbell-Hausdorff formula it follows that the spacetime vector field $X^{(1)\mu}_{a\ldots}(x)$ transforms as

$$\big(X^{(1)\mathcal{E}}_{a\ldots}\big)^\mu = \big(\exp(\pounds_{\mathcal{E}}) X^{(1)}_{a\ldots}\big)^\mu + \ldots\ . \tag{7.19}$$

This should be compared with the diff-transformation of a vector field

$$V'^\mu(x') = \partial_\nu \zeta^\mu(x)\, V^\nu(x) \qquad , \qquad x'^\mu = \zeta^\mu(x) \tag{7.20}$$

The large MHS transformation (7.19) is a diffeomorphism, where the connection between parameter fields seems to be given by

$$\begin{aligned}\zeta^\mu(x) - x^\mu =& \sum_{r=0}^{\infty} \frac{(\mathcal{E}(x)\cdot\partial)^r}{(r+1)!}\,\mathcal{E}^\mu(x) \\ =& \frac{e^{\mathcal{E}(x)\cdot\partial}-1}{\mathcal{E}(x)\cdot\partial}\,\mathcal{E}^\mu(x)\ .\end{aligned} \tag{7.21}$$

We have checked this relation up to the quartic order [88, 89]. We see that large MHS transformations in the spin-2 sector are indeed finite diffeomorphisms, but that the naturally defined parameters of the two descriptions are related in a complicated way given by (7.21). Let us now analyze the metric, whose low-spin components in the MHS framework are naturally obtained from (2.90). The result is

$$g_{(0)}(x) = \frac{1}{2} A_a(x)\, A^a(x) + \frac{1}{2}\partial_\nu E_a{}^\mu(x)\, \partial_\mu E^{a\nu}(x) + \ldots \tag{7.22}$$

$$g^\mu_{(1)}(x) = E^{a\mu}(x)\, A_a(x) + \ldots \tag{7.23}$$

$$g^{\mu\nu}_{(2)}(x) = E^{a\mu}(x)\, E_a{}^\nu(x) + \ldots\ . \tag{7.24}$$

Relations (7.23)-(7.24) confirm the identification of

$$A^\mu(x) \equiv E^{a\mu}(x)\, A_a(x) \tag{7.25}$$

as a $U(1)$ vector potential, $E_a{}^\mu(x)$ as a vielbein and

$$g^{\mu\nu}(x) \equiv E^{a\mu}(x)\, E_a{}^\nu(x) \tag{7.26}$$

as the (inverse) metric tensor. The terms on the right hand side of (7.22) are responsible for producing the seagull interaction terms when a Klein-Gordon matter field is minimally coupled to the MHS field.

To explore a possible induced geometric picture, we define a linear connection by demanding compatibility between the linear part in the expansion of the MHS covariant derivative (7.8) and the induced geometric covariant derivative

$$(\mathcal{D}^{\star}_a V)^{\mu}_{(1)}(x) = \left(E_a{}^{(1)\nu}\partial_\nu V^{\mu}_{(1)} - V^{\nu}_{(1)}\partial_\nu E_a{}^{(1)\mu}\right) \equiv E_a{}^{(1)\nu}\nabla_\nu V^{\mu} \,. \tag{7.27}$$

from which it follows that the induced covariant derivative of a vector field should be given by

$$(\nabla_{E_a} V)^{\mu} \equiv E_a{}^{\nu}\nabla_\nu V^{\mu} = \left(\pounds_{E_a} V\right)^{\mu} \,. \tag{7.28}$$

Multiplying by $E^a{}_\mu(x)$ we finally obtain

$$\begin{aligned}\nabla_\nu V^{\mu} =& E^a{}_\nu\left(\pounds_{E_a} V\right)^{\mu} \\ =& \partial_\nu V^{\mu} + E_a{}^{\mu}\partial_\rho E^a{}_\nu \, V^{\rho} \,.\end{aligned} \tag{7.29}$$

This means that the MHS symmetry induces the following linear connection

$$\Gamma^{\mu}{}_{\rho\nu} = E_a{}^{\mu}\partial_\rho E^a{}_\nu = -E^a{}_\nu\partial_\rho E_a{}^{\mu} \,. \tag{7.30}$$

The obtained linear connection is very much different from the Levi-Civita connection. For one, the torsion tensor is generally non-vanishing, as it can explicitly be checked that

$$\begin{aligned}T^{\mu}{}_{\rho\nu} =& \Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\mu}{}_{\rho\nu} \\ =& \xi^{\mu}{}_{\rho\nu} \equiv E^a{}_\rho E^b{}_\nu \xi^{\mu}{}_{ab} \,,\end{aligned} \tag{7.31}$$

where $\xi^{\mu}{}_{ab}(x)$ is

$$\xi^{\mu}{}_{ab} = \left(\pounds_{E_a} E_b\right)^{\mu} = \xi^{c}{}_{ab} E_c{}^{\mu} \,. \tag{7.32}$$

$\xi^{c}{}_{ab}(x)$ are known as coefficients of anholonomy. As a consistency check, let us calculate the $n = 1$ component of the HS torsion. It is easy to show that it is given by

$$T^{(1)\mu}_{ab}(x) = \xi^{\mu}{}_{ab}(x) + \ldots \tag{7.33}$$

which is consistent with (7.31).

Also, the linear connection (7.30) is not metric compatible, the nonmetricity tensor being

$$Q_\rho{}^{\mu\nu} \equiv \nabla_\rho \, g^{\mu\nu} = T^{\mu}{}_{\rho\sigma} \, g^{\rho\nu} + T^{\nu}{}_{\rho\sigma} \, g^{\rho\mu} = T^{\mu\nu}{}_{\sigma} + T^{\nu\mu}{}_{\sigma} \tag{7.34}$$

which is generally non-vanishing. Note that the nonmetricity tensor is not independent but is fully (algebraically) expressible in terms of torsion. The same is true for the

Riemann tensor, given for a general linear connection as (see e.g. [60] or [90] for a general exposition)

$$R^{\rho}{}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma^{\rho}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\eta\sigma}\Gamma^{\eta}{}_{\mu\nu} - \Gamma^{\rho}{}_{\eta\nu}\Gamma^{\eta}{}_{\mu\sigma} \tag{7.35}$$

for which it can be shown that it can be expressed in terms of the torsion tensor and its covariant derivatives

$$R^{\rho}{}_{\theta\mu\nu} = \nabla_{\mu}T^{\rho}{}_{\theta\nu} - \nabla_{\nu}T^{\rho}{}_{\theta\mu} - T^{\rho}{}_{\alpha\mu}T^{\alpha}{}_{\theta\nu} + T^{\alpha}{}_{\theta\mu}T^{\rho}{}_{\alpha\nu} - T^{\alpha}{}_{\mu\nu}T^{\rho}{}_{\theta\alpha}\,. \tag{7.36}$$

Note that the usual algebraic symmetries of the Riemann tensor for a Riemannian connection are not all present if torsion is non-vanishing [60]. Let us also calculate the spin connection in the geometry induced by the MHS construction. The simplest way to find it is to use

$$A^{a}{}_{b\mu} = E^{a}{}_{\nu}\,\nabla_{\mu}E_{b}{}^{\nu}\;. \tag{7.37}$$

Using (7.30) we get

$$A^{a}{}_{b\mu} = E^{a}{}_{\nu}E_{b}{}^{\rho}\,T^{\nu}{}_{\mu\rho} = T^{a}{}_{\mu b}\;. \tag{7.38}$$

We see that $A_{ab\mu}$ is not antisymmetric in its first two indices, which is a manifestation of metric incompatibility. Again, we see that the induced spin connection is fully determined by the torsion.

7.1.1 Connection to teleparallelism

We have seen that the induced spacetime geometry found in the $s = 2$ ($n = 1$) sector of the MHS theory seems rather unusual. The linear connection is metric-incompatible, and both the torsion and the Riemann tensor are non-vanishing. It is in fact closely related to teleparallel geometry. The key observation is that there is only one independent fundamental tensor, the torsion, and all others are expressible in terms of it.

Let us first briefly review the concept of distant parallelism or teleparallelism.[1] Let us assume that a differentiable manifold is equipped with a linear connection Γ_{+}, which is not symmetric. Teleparallelism is a requirement on the linear connection that there exists a frame of vector fields (an inertial frame) $E_{a}{}^{\mu}(x)$ that globally satisfies

$$\nabla^{+}_{\mu}E_{a}{}^{\sigma} \equiv \partial_{\mu}E_{a}{}^{\sigma} + \Gamma^{\sigma}_{+\rho\mu}E_{a}{}^{\rho} = 0\;. \tag{7.39}$$

[1]For a detailed exposition of teleparallel geometry and gravity see the book [91].

From the definition of the covariant derivative it follows that the linear connection is given by

$$\Gamma^{\sigma}_{+\rho\mu} = E_a{}^{\sigma}\partial_\mu E^a{}_\rho = -E^a{}_\rho\,\partial_\mu E_a{}^{\sigma} \tag{7.40}$$

which is known as the Weitzenböck connection. If the metric is defined by taking the inertial frame as the vielbein

$$g^{\mu\nu} = \eta^{ab}E_a{}^{\mu}E_b{}^{\nu} \tag{7.41}$$

then from (7.39) it obviously follows that the Weitzenböck connection is metric compatible

$$\nabla^{+}_{\rho}g_{\mu\nu} = 0 \tag{7.42}$$

An outstanding property of the Weitzenböck connection is that its corresponding spin (Lorentz) connection is vanishing

$$\mathscr{A}^{a}_{+b\mu} = E^b{}_\sigma\nabla_\mu E_a{}^{\sigma} = 0 \tag{7.43}$$

for inertial frames. Inertial frames are related to one another through global Lorentz transformations,

$$E_a{}^{\mu}(x) \to \Lambda_a{}^{b}E_b{}^{\mu}(x)\ . \tag{7.44}$$

Note that by performing a *local* Lorentz transformation

$$E_a{}^{\mu}(x) \to \Lambda_a{}^{b}(x)E_b{}^{\mu}(x) \qquad , \qquad \partial_\mu\Lambda_a{}^{b} \neq 0 \tag{7.45}$$

one passes to a non-inertial frame which does not satisfy (7.39). As a consequence the spin connection in the transformed frame is non-vanishing but still trivial (i.e. flat),

$$\mathscr{A}^{a}_{+b\mu} \to \Lambda_b{}^{c}\partial_\mu\Lambda_c{}^{a}\ . \tag{7.46}$$

It follows directly from (7.43) that the Riemann tensor also vanishes

$$R^{a}_{+b\mu\nu} = 0 \tag{7.47}$$

which, as a consistency check, one could also show using the Weitzenböck linear connection. This means that, beside the metric, the only nontrivial fundamental tensor in teleparallel geometry is torsion, which is given (using the inertial frame) by

$$\begin{aligned} T^{\mu}_{+\nu\rho} \equiv& \Gamma^{\mu}_{+\rho\nu} - \Gamma^{\mu}_{+\nu\rho} \\ =& -\xi^{\mu}{}_{\nu\rho} \end{aligned} \tag{7.48}$$

where the anholonomy ξ was defined in (7.32).

The simplest Lagrangians of teleparallel gravity theories are of the form [92]

$$S_{\mathrm{tg}} = \int d^d x \, E \left(c_1 \, T^{\rho}_{+\mu\nu} T_{\rho}^{+\mu\nu} + c_2 \, T^{\rho}_{+\mu\nu} T^{\nu\mu}_{+\ \ \rho} + c_3 \, T^{\rho}_{+\mu\rho} T^{+\ \nu}_{\nu\mu} \right) \tag{7.49}$$

It can be shown that if one takes

$$c_1 = \frac{1}{4} \qquad , \qquad c_2 = \frac{1}{2} \qquad , \qquad c_3 = -1 \tag{7.50}$$

then (7.49) becomes equal, up to a boundary term, to the Einstein-Hilbert action. The theory based on this action is usually called the teleparallel equivalent of General Relativity (TEGR). One of the advantages of the teleparallel formulation (over the Einstein-Hilbert one) is having a manifestly diff-covariant Lagrangian which contains derivative terms only up to first order. Teleparallel gravity theories were first studied by Albert Einstein already in the 1920's [93].

Let us now finally connect the geometry emerging from the MHS symmetry with teleparallel geometry. Comparing their respective linear connections, (7.30) and (7.40), we see that

$$\Gamma^{\mu}{}_{\nu\rho} = \Gamma^{\mu}_{+\rho\nu} \tag{7.51}$$

i.e., our linear connection is the *opposite* of the teleparallel one. It is then not strange that torsions are related by

$$T^{\mu}{}_{\nu\rho} = -T^{\mu}_{+\nu\rho} \, . \tag{7.52}$$

This means that the covariant derivative induced by MHS symmetry can be written in terms of the covariant derivative of teleparallel geometry

$$\nabla = \nabla_{+} - T_{+} \, . \tag{7.53}$$

Using (7.52) and (7.53) we can express any covariant expression in the teleparallel geometry as a covariant expression in the opposite of teleparallel geometry. As a special case, it means that a manifestly covariant EoM in the emergent MHS geometry can be expressed as a manifestly covariant EoM in teleparallel geometry, and vice versa. This will be important below in the discussion of $s = 2$ sector of EoM in the MHSYM model.

Teleparallel gravity can be obtained by gauging the group isometric to the group of spacetime translations [94, 95, 96]. As the global MHS transformations have such a subgroup ($n = 1$ sector), it is not surprising that there is a connection between the MHS theory and teleparallel gravity.

7.2 MHSYM model in the $s \leq 2$ sector

In view of the preceding discussion on the induced spacetime geometry in the MHS construction, it is interesting to study the $s \leq 2$ sector of EoM of the MHSYM model. We put (7.4) into the MHSYM EoM

$$\mathcal{D}^{\star}_{a} T^{ab}(x,u) \doteq 0 \tag{7.54}$$

and take into consideration only the purely $s \leq 2$ components. The $s = 1$ component of the EoM is given by

$$0 = E_b{}^{\nu} \partial_\nu F^{ba} - \xi^{\nu ba} \partial_\nu A_b + \dots \tag{7.55}$$

where $\mathbf{F}$ is the 2-form field strength of the spin-1 $U(1)$ spacetime vector potential 1-form $\mathbf{A}$, i.e.

$$\begin{aligned} F_{ab} =& E_a{}^{\mu} E_b{}^{\nu} F_{\mu\nu} = E_a{}^{\mu} E_b{}^{\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ =& E_a{}^{\nu} \partial_\nu A_b - E_b{}^{\nu} \partial_\nu A_a + A_c \, \xi^{c}{}_{ab} \, . \end{aligned} \tag{7.56}$$

The $s = 2$ component of the EoM is

$$0 = E_b{}^{\nu} \partial_\nu \xi^{\mu ba} - \xi^{\nu ba} \partial_\nu E_b{}^{\mu} + \dots \; . \tag{7.57}$$

After some manipulation we can rewrite this equation in the equivalent form

$$0 = E_c{}^{\nu} \partial_\nu \xi^{abc} - \xi^{a}{}_{cd} \, \xi^{cdb} + \dots \; . \tag{7.58}$$

Also, using (7.58) and (7.56) we can write the $s = 1$ EoM component (7.55) as

$$0 = E_b{}^{\nu} \partial_\nu F^{ba} - \xi^{bca} F_{bc} + \dots \tag{7.59}$$

which is manifestly $U(1)$-gauge invariant.

Terms denoted by ellipses, and also complete $s > 2$ components of EoM, vanish if all $s > 2$ components of the MHS vielbein vanish. This is a consequence of the fact that the MHSYM theory can be consistently truncated to the low-spin ($s \leq 2$) sector at the level of EoM (though we keep in mind limited applicability of this approach for a complete description of the MHSYM theory).

Let us rewrite the low-spin EoM (7.58-7.59) within the framework of teleparallel geometry by using (7.52) and the fact that in teleparallel gravity the Lorentz connection is vanishing in inertial frames so the Lorentz covariant derivative is simply the coordinate derivative

$$\mathscr{D}_\mu^+ = \partial_\mu \qquad \Longrightarrow \qquad \mathscr{D}_a^+ = E_a{}^\mu \partial_\mu \,. \tag{7.60}$$

Using all of this we can write (7.59) as

$$\mathscr{D}_b^+ F^{ba} + T_+^{bca} F_{bc} = 0 \tag{7.61}$$

which is now fully diff- and $U(1)$ covariant. Similarly, (7.58) becomes

$$\mathscr{D}_c^+ T_+^{abc} + T^a_{+cd} T_+^{cdb} = 0 \tag{7.62}$$

which is manifestly diff-covariant. We stress that in this form all objects in (7.61) and (7.62) should be calculated using the Weitzenböck connection. In other words, they are formally written within the realm of teleparallel gravity, and not the geometry induced by the MHS symmetry.

The equation (7.62) was first written by Albert Einstein in 1929, with a motivation to unify electromagnetism with gravity [93].[2]. Einstein observed that it is not possible to write a diff-covariant action which produces (7.62) as its EoM inside the realm of the teleparallel geometry, because the left hand side is not covariantly conserved (in particular it does not belong to the class of theories defined in (7.49)). It is amusing that we obtained EoM (7.62) from an action principle by truncating the MHSYM theory.

7.3 Classical vacuum solutions

The complete (non-linear) EoM of the MHSYM theory with no source is

$$\mathcal{D}_a^\star T^{ab}(x,u) \doteq 0 \tag{7.63}$$

or explicitly

$$[e_b(x,u) \overset{\star}{,} [e^a(x,u) \overset{\star}{,} e^b(x,u)]] = 0\,. \tag{7.64}$$

The simplest solution of the Yang-Mills case, as we have seen in (2.77) is given by the Minkowski background

$$e_a(x,u) = u_a = \delta_a^\mu u_\mu \,. \tag{7.65}$$

[2]For this reason he added by hand one more equation to EoM in an attempt to project out unwanted degrees of freedom. Already in 1930 he abandoned this attempt.

We can use the the Minkowski background as a motivation to examine more general vacuum solutions and seek for them in the following form

$$e_a(x,u) = E_a{}^{\mu}(x)u_{\mu}\,. \tag{7.66}$$

As we've argued above, the equations of motion and the gauge symmetry are consistent if we restrict our attention to fields linear in u_{μ}, which we already dubbed the "spin 2" sector, however, our motivation here is more inclined towards exact vacuum solutions than truncated configurations. We will examine possible solutions in the pure Yang-Mills case as well as the case with a non-vanishing cosmological constant as introduced in chapter 5.

7.3.1 Spherical background solution for MHS Yang-Mills theory

We use the proposed form for the background solution (7.66) and insert it into the MHSYM EoM (7.64). Only the linear terms in u_{μ} survive and we arrive at

$$E_a{}^{\mu}\partial_{\mu}\xi^{\nu ab} - \xi^{\mu ab}\partial_{\mu}E_a{}^{\nu} = 0 \tag{7.67}$$

where $\xi^{\mu ab} = E^{a\nu}\partial_{\nu}E^{b\mu} - E^{b\nu}\partial_{\nu}E^{a\mu}$ was already introduced in (7.32). Latin indices are raised/lowered with the Minkowski metric η_{ab}. We will seek for a spherically symmetric solution, and for that reason we introduce an Ansatz:

$$E_a{}^{\mu}(t,x,y,z) = \mathrm{diag}(\alpha(r),\beta(r),\beta(r),\beta(r)) \tag{7.68}$$

Where $r = \sqrt{x^2+y^2+z^2}$. It is straightforward to insert (7.68) into (7.67) and calculate the necessary equations. Of the 16 field equations, the non-trivial are:

$$\beta(r)\left(-\alpha''(r) - \frac{2\alpha'(r)}{r}\right) - \alpha'(r)\beta'(r) = 0 \tag{7.69}$$

$$\beta(r)\left(\beta'(r)\left(2x^2+y^2+z^2\right) + r\left(y^2+z^2\right)\beta''(r)\right) + r\left(y^2+z^2\right)\beta'(r)^2 = 0 \tag{7.70}$$

$$\beta(r)\left(\beta'(r)\left(x^2+2y^2+z^2\right) + r\left(x^2+z^2\right)\beta''(r)\right) + r\left(x^2+z^2\right)\beta'(r)^2 = 0 \tag{7.71}$$

$$\beta(r)\left(\beta'(r)\left(x^2+y^2+2z^2\right) + r\left(x^2+y^2\right)\beta''(r)\right) + r\left(x^2+y^2\right)\beta'(r)^2 = 0 \tag{7.72}$$

$$\beta(r)\left(\beta'(r) - r\beta''(r)\right) - r\beta'(r)^2 = 0 \tag{7.73}$$

where prime stands for the derivative against the argument of a function. Equations (7.70-7.72) lead to the condition

$$\beta(r)\left(2\beta'(r) + r\beta''(r)\right) + r\beta'(r)^2 = 0 \tag{7.74}$$

which we can insert into (7.73) to obtain a simple equation for $\beta(r)$.

$$\beta(r)\beta'(r) = 0\,.$$

Thus $\beta(r) = B$ is a constant. Upon using this solution and inserting it into (7.69) we finally obtain

$$\alpha(r) = D + \frac{C}{r} \tag{7.75}$$

where C, D are integration constants.

The geometric interpretation developed in the previous section enables us to use the obtained solution and compute the induced metric (7.41)

$$ds^2 = -\frac{1}{\left(D + \frac{C}{r}\right)^2}dt^2 + \frac{1}{B^2}(dx^2 + dy^2 + dz^2)\,. \tag{7.76}$$

We can calculate the curvature tensors following the definitions (7.30-7.36). The connection is fundamentally not free of torsion, and one has to be careful not to use expressions ordinarily useful in general relativity. As an interesting result, we report the value of the Ricci scalar

$$R = \frac{4B^2C^2}{r^2(Dr + C)^2}\,. \tag{7.77}$$

We can see that the singularities $r = 0, -\frac{C}{D}$ are not coordinate artifacts.

7.3.2 Spherical background solution for MHS Yang-Mills theory with a cosmological constant

In the more general MHS model which we explored in chapter 5, the equation of motion contains an additional linear term in the MHS vielbein

$$[e_b(x,u) \mathbin{\overset{\star}{,}} [e^a(x,u) \mathbin{\overset{\star}{,}} e^b(x,u)]] = \lambda_1 e^b(x,u) \tag{7.78}$$

The theory is changed substantially with $\lambda_1 \neq 0$ with respect to the pure MHSYM, as now the Minkowski background is no longer a solution. We can again consider finding a background solution in the form of (7.66), and the feature of having only linear terms in u_μ remains. We arrive at the field equations

$$E_a{}^\mu \partial_\mu \xi^{\nu ab} - \xi^{\mu ab}\partial_\mu E_a{}^\nu = \lambda_1 E^{b\nu}\,, \tag{7.79}$$

where again $\xi^{\mu ab} = E^{a\nu}\partial_\nu E^{b\mu} - E^{b\nu}\partial_\nu E^{a\mu}$. A spherically symmetric Ansatz as (7.68) can again be used, and with it the non-trivial field equations are obtained as

$$\beta(r)\left(-\alpha'(r)\beta'(r) - \frac{2}{r}\beta(r)\alpha'(r) - \beta(r)\alpha''(r)\right) = -4\lambda_1\alpha(r) \tag{7.80}$$

$$(y^2+z^2)r\beta'(r)^2 + \beta(r)((2x^2+y^2+z^2)\beta'(r) + r(y^2+z^2)\beta''(r)) = 4r^3\lambda_1 \tag{7.81}$$

$$(x^2+z^2)r\beta'(r)^2 + \beta(r)((x^2+2y^2+z^2)\beta'(r) + r(x^2+z^2)\beta''(r)) = 4r^3\lambda_1 \tag{7.82}$$

$$(x^2+y^2)r\beta'(r)^2 + \beta(r)((x^2+y^2+2z^2)\beta'(r) + r(x^2+y^2)\beta''(r)) = 4r^3\lambda_1 \tag{7.83}$$

$$-r\beta'(r)^2 + \beta(r)(\beta'(r) - r\beta''(r)) = 0 \tag{7.84}$$

We can proceed to find a solution in the same way. The combination of (7.81 - 7.83) leads us to the conclusion

$$2r^3\beta'(r)^2 + \beta(r)\left(\beta'(r)4r^2 + 2r^3\beta''(r)\right) = 12r^3\lambda_1\,. \tag{7.85}$$

$$r\beta'(r)^2 + \beta(r)\left(2\beta'(r) + r\beta''(r)\right) = 6r\lambda_1\,. \tag{7.86}$$

Along with (7.84) we can obtain a differential equation for $\beta(r)$

$$\beta(r)\beta'(r) = 2r\lambda_1\,. \tag{7.87}$$

We solve it and find the solution for

$$\beta^2(r) = 2\lambda_1(r^2 - c_1)\,. \tag{7.88}$$

Now we insert this solution into (7.80) and find a differential equation for $\alpha(r)$

$$\alpha''(r)(r^2 - c_1) + \alpha'(r)(3r - 2\frac{c_1}{r}) - 2\alpha(r) = 0\,. \tag{7.89}$$

The solution to this differential equation is not too difficult to find, and it is given by

$$\alpha(r) = \frac{d_1}{r}\cos\left(\sqrt{3}\arccos\left(\frac{r}{\sqrt{c_1}}\right)\right) + \frac{d_2}{r}\sin\left(2\sqrt{3}\arcsin\left(\sqrt{\frac{1}{\sqrt{2}} - \frac{r}{\sqrt{2c_1}}}\right)\right) \tag{7.90}$$

We can also examine the special case when the integration constant c_1 vanishes. Then it is obvious from (7.88) that λ_1 can be only positive. The differential equation for $\alpha(r)$ becomes simpler

$$r^2\alpha''(r) + 3r\alpha'(r) - 2\alpha(r) = 0\,. \tag{7.91}$$

The solution for $\alpha(r)$ is now given by

$$\alpha(r) = d_3\,r^{\sqrt{3}-1} + d_4 r^{-\sqrt{3}-1}\,. \tag{7.92}$$

With the solutions for $\alpha(r)$ and $\beta(r)$ it is straightforward to calculate the curvature tensors. The explicit expression for e.g. the Ricci scalar in the $c_1 = 0$ case is

$$R = -2\lambda_1 \left(\frac{2d_4 \left(\left(5\sqrt{3} - 24\right) d_3 r^{2\sqrt{3}} + 5\sqrt{3} d_4 \right)}{\left(d_3 r^{2\sqrt{3}} + d_4\right)^2} - 5\left(1 + \sqrt{3}\right) \right) \tag{7.93}$$

Specifically, there are two cases in which the Ricci scalar becomes constant;

$$R\Big|_{d_3=0} = -\lambda_1 \, 10(\sqrt{3} - 1) \tag{7.94}$$

$$R\Big|_{d_4=0} = \lambda_1 \, 10(\sqrt{3} + 1)\,. \tag{7.95}$$

The spacetime element in these two specific cases becomes

$$ds^2 = -\frac{1}{(d_3)^2 r^{2\sqrt{3}-2}} dt^2 + \frac{1}{(2\lambda_1 r^2)} (dx^2 + dy^2 + dz^2) \tag{7.96}$$

$$ds^2 = -\frac{r^{2\sqrt{3}+2}}{(d_4)^2} dt^2 + \frac{1}{(2\lambda_1 r^2)} (dx^2 + dy^2 + dz^2) \tag{7.97}$$

We can conclude that the addition of the "cosmological constant" term on the action for the MHS model admits background solutions whose scalar curvature is constant, a feature shared with spaces of maximal symmetry.

Chapter 8

Conclusion and outlook

We have realized a gauging procedure of the higher-spin symmetries (2.2)

$$\delta_\varepsilon \phi(x) = \sum_{n=0}^{\infty} (-i)^{n+1} \varepsilon^{\mu_1 \dots \mu_n} \, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x) \, .$$

The appearance of the Moyal product in our construction can be seen as a way to close the algebra of symmetries in case we promote the variation parameters to functions on spacetime. Then, the gauge algebra is given by a Moyal bracket of functions on the master space (2.34)

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{i[\varepsilon_1 \stackrel{\star}{,} \varepsilon_2]} \, .$$

A complete realization of our gauge field off-shell was done on a master space, on which we have formulated a Yang-Mills-like theory. Differently from a deformation program of finding interacting theories from known free theories, we have relied on symmetries of our gauge field and identified the general MHS covariant construction. This allowed us to see that there can be different phases of the theory, of which only one corresponds to the Yang-Mills like construction we started with.

The MHSYM model comes with an action principle, it is perturbatively stable and admits a description in terms of the L_∞ algebra. These are qualitative features that we expect from a well defined theory.

We moved on to an analysis of the particle spectrum of the MHSYM theory. One direction of this analysis required an explicit construction of a new representation of the Lorentz group on the space of multi-dimensional Hermite functions. These results can be used even without the context of the MHS theory. A possible use could be in the context of computer graphics [97], since we have provided explicit expressions for 2D and 3D rotations of Hermite expansions.

An analysis of the particle spectrum based on the differential equations posed by the little group for massless particles lead us to the conclusion that our theory contains degrees of freedom of infinite spin, and we have identified an on-shell basis of polarization functions.

The MHS symmetry and the master space can be explored in further directions. We have shown an analysis of the conservation laws based on the MHS symmetry which led to an expression of an infinite tower of conserved currents. Reliance on the symmetries also enabled us to put forward new candidates for theories based on these structures.

Within the framework of the formalism, we were also able to describe matter and show how it couples to the MHS gauge potential minimally. We calculated the simplest scattering amplitudes in the lowest order in the perturbation theory and found that scattering of simple matter is allowed only for sets of momenta being equal in the final as in the initial state. In the case of master space matter in the fundamental representation the scattering amplitude displayed a softer behaviour with respect to quantum electrodynamics.

Finally, we have focused on a possible geometric interpretation of the low-spin sector of the theory. Although a Taylor expansion in the auxiliary space is not completely rigorous (the component fields are then never independent), we were still able to extract valuable geometric information about the theory, albeit working only on-shell. The reminiscence to teleparallel geometry was not so surprising, as translations form a subgroup of the gauged symmetries (2.2).

Possible background solutions of the full theory do not necessarily follow the same convergence properties as the potential fields (e.g. the Minkowski vacuum $e_a = u_a$ grows linearly in the auxiliary space), so we have analyzed additional background solutions of the similar form, both in the MSHYM case, and in the case with a cosmological constant term present in the action. Since then the equations for the background solutions were of the spin-2 form, results from the low-spin sector were applied to this goal.

The MHS theory in its current form is not a finalized model. As mentioned above, the amplitude for the tree-level scattering of minimal matter contains delta functions in the momenta. To find the cross-section, the amplitude would have to be squared and there would appear a factor of $\delta^{(d)}(0)$. This divergent factor could be interpreted as the volume of the auxiliary space. Within the calculation, it is noticeable that the origin of the delta functions is the completeness relation of the basis functions, or in other words, in the sum over the entire particle configuration space. An analogous problem appears when finding inclusive cross-sections with the outer legs representing the MHS particles.

A future perspective would then be to find a way of restricting the configuration space in some way; ideally to contain a single irreducible representation of the Poincaré group. In this regard, it might prove interesting to still rely on the MHS symmetries and the master space, and try to modify the dynamics of the theory in a way to make contact with [73, 74, 75, 76]. One could also consider restricting the auxiliary space in some manner which would reduce the configuration space as a consequence, but care must be taken that the restriction is done in an MHS covariant manner.

The MHS structures are very rich and we hope they will have a say in further research in higher spin and infinite spin particles.

Appendix A

Mathematical definitions and results

A.1 Moyal star product

We give the definition and the most important properties of the Moyal star product [98, 99] relevant to this work. Detailed expositions are available in [100, 101, 102].

For functions $a(x,u), b(x,u)$ defined on $\mathbb{R}^{2n}$, the Moyal star product is defined as

$$a(x,u) \star b(x,u) = a(x,u)\ \exp\left[\frac{i}{2}\left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_u - \overrightarrow{\partial}_x \cdot \overleftarrow{\partial}_u\right)\right] b(x,u) \tag{A.1}$$

For general functions, a Moyal product includes an infinite number of derivatives both over x and u, and can be seen as a deformation of the ordinary multiplication rule

$$a(x,u) \star b(x,u) = a(x,u)b(x,u) + \frac{i}{2}\left(\frac{\partial a(x,u)}{\partial x^\mu}\frac{\partial b(x,u)}{\partial u_\mu} - \frac{\partial a(x,u)}{\partial u_\mu}\frac{\partial b(x,u)}{\partial x^\mu}\right) + \cdots \tag{A.2}$$

For example,

$$x^\mu \star x^\nu = x^\mu x^\nu \tag{A.3}$$

$$x^\mu \star u_\nu = x^\mu u_\nu + \frac{i}{2}\delta^\mu_\nu \tag{A.4}$$

$$u_\mu \star u_\nu = u_\mu u_\nu \tag{A.5}$$

$$u_\nu \star x^\mu = u_\nu x^\mu - \frac{i}{2}\delta^\mu_\nu\,. \tag{A.6}$$

The Moyal commutator and anticommutator are defined naturally as

$$[a(x,u) \overset{\star}{,} b(x,u)] \equiv a(x,u) \star b(x,u) - b(x,u) \star a(x,u) \tag{A.7}$$

$$\{a(x,u) \overset{\star}{,} b(x,u)\} \equiv a(x,u) \star b(x,u) + b(x,u) \star a(x,u)\,. \tag{A.8}$$

The Moyal product is

- Hermitean under complex conjugation

$$(a(x,u) \star b(x,u))^* = b(x,u)^* \star a(x,u)^* \; . \tag{A.9}$$

- Associative

$$\big(a(x,u) \star b(x,u)\big) \star c(x,u) = a(x,u) \star \big(b(x,u) \star c(x,u)\big) \tag{A.10}$$

- Obeys the Jacobi identity

$$[a \overset{\star}{,} [b \overset{\star}{,} c]] + [c \overset{\star}{,} [a \overset{\star}{,} b]] + [b \overset{\star}{,} [c \overset{\star}{,} a]] = 0 \tag{A.11}$$

- Follow's the Leibniz rule

$$[a \overset{\star}{,} b \star c] = [a \overset{\star}{,} b] \star c + b \star [a \overset{\star}{,} c] \; . \tag{A.12}$$

For a class of functions with well behaved fall-of conditions the Moyal product can be calculated in the integral form

$$a(x,u) \star b(x,u) = \int d^d y \, d^d z \, \frac{d^d v}{(2\pi)^d} \frac{d^d w}{(2\pi)^d} e^{i(yw - zv)} a(x + \frac{y}{2}, u + v) b(x + \frac{z}{2}, u + w) \, . \tag{A.13}$$

A very useful and convenient way to make calculations with the Moyal product is by promoting coordinates to operators, of which we note one possible way:

$$a(x,u) \star b(x,u) = a(x, \mathbf{u}) b(x, \mathbf{u}^\dagger) \tag{A.14}$$

with

$$\mathbf{u} = u - \frac{i}{2} \vec{\partial}_x, \qquad \mathbf{u}^\dagger = u + \frac{i}{2} \overleftarrow{\partial}_x \tag{A.15}$$

Moyal commutator of real functions is purely imaginary, while the Moyal anticommutator of real functions is real.

Under integration it satisfies the adjoint property

$$\begin{aligned} \int d^d x \, d^d u \, \big(a(x,u) \star b(x,u)\big) c(x,u) &= \int d^d x \, d^d u \, a(x,u) \big(b(x,u) \star c(x,u)\big) \\ &= \int d^d x \, d^d u \, b(x,u) \big(c(x,u) \star a(x,u)\big) \end{aligned} \tag{A.16}$$

where a, b and c are square-integrable functions on the master space. If we put $c(x,u) = 1$ we obtain

$$\int d^d x \, d^d u \, a(x,u) \star b(x,u) = \int d^d x \, d^d u \, a(x,u) \, b(x,u) + (\text{boundary terms}) \; . \tag{A.17}$$

The boundary terms are a sum of total derivatives in x and u spaces.

$$\partial^x_a(C^a(x,u)) + \partial^a_u(D_a(x,u))\,. \tag{A.18}$$

The $\star$ - exponential is defined as

$$e_\star^{a(x,u)} = \sum_{n=0}^{\infty} \frac{1}{n!}\, a(x,u)^{\star n} \tag{A.19}$$

where $a(x,u)^{\star n}$ is the Moyal product with n factors of $a(x,u)$

$$a(x,u)^{\star n} = a(x,u) \star a(x,u) \star \ldots \star a(x,u)\ . \tag{A.20}$$

A.2 Weyl-Wigner map

The Weyl-Wigner map (or correspondence) is an invertible integral transformation between functions defined on $\mathbb{R}^{2n}$ and operators on a Hilbert space. First appearance is in [103, 104] while reviews can be found in [100, 105]. For use in physics of higher spins, see [106] and appendices of [44, 43].

The Wigner map takes an operator $\hat{F}$ acting on the Hilbert space $\mathcal{H}$ and outputs a function $f(x,u)$ (dubbed symbol of the operator) on a phase space (master space) spanned by $\{x^a, u_a\}$.

$$\mathcal{W}[\hat{F}] = f(x,u) = \int d^dq\, \langle x - \frac{q}{2}|\hat{F}(\hat{X},\hat{U})|x + \frac{q}{2}\rangle e^{iq\cdot u} \tag{A.21}$$

The inverse is named the Weyl map and it is given by

$$\mathcal{W}^{-1}[f(x,u)] = \hat{F}(\hat{X},\hat{U}) = \int d^dx\, d^dy \frac{d^dk}{(2\pi)^d}\frac{d^du}{(2\pi)^d}\, f(x,u)\, e^{ik\cdot(x-\widehat{X})-iy\cdot(p-\widehat{P})} \tag{A.22}$$

The Weyl-Wigner map relates complex conjugation of symbols to Hermitean conjugation of operators

$$f^*(x,u) \leftrightarrow \hat{F}^\dagger\,. \tag{A.23}$$

The trace of an operator is given by the integral of the corresponding function over the master space

$$\mathrm{Tr}(\hat{F}) = \int d^dx\, \frac{d^du}{(2\pi)^d}\, f(x,u)\ . \tag{A.24}$$

A product of operators on the Hilbert space in the Weyl (completely symmetric) ordering becomes the Moyal star product of functions on the master space

$$\mathcal{W}[\hat{F}_1\, \cdot \hat{F}_2] = f_1(x,u) \star f_2(x,u)\,, \tag{A.25}$$

and vice-versa

$$\mathcal{W}^{-1}[f_1(x,u) \star f_2(x,u)] = \hat{F}_1 \cdot \hat{F}_2 \,. \tag{A.26}$$

The trace of a product of operators is then

$$\mathrm{Tr}[\hat{F}_1 \cdot \hat{F}_2] = \int d^d x \, \frac{d^d u}{(2\pi)^d} \, f_1(x,u) \star f_2(x,u) \,. \tag{A.27}$$

A.3 From Lie to L_∞

The identification of such a rich mathematical structure as is the L_∞ algebra (also named strongly homotopy Lie algebra) can seem almost miraculous, so we will try to define it through a gradual generalization of Lie algebras in the following way:

Lie alg. $\rightarrow$ graded Lie alg. $\rightarrow$differential graded Lie alg. $\rightarrow$ diff. grad. homotopy Lie. alg
$\rightarrow$strongly homotopy Lie algebra

A Lie Algebra is a vector space together with a bilinear antisymmetric product named the Lie bracket that satisfies the Jacobi identity. Let us name the vector space $V = X_0$ with elements $x_1, x_2, ...$ and denote the Lie bracket as

$$[x_1, x_2] \equiv \ell_2(x_1, x_2) = -\ell_2(x_2, x_1); \quad \ell_2 : X_0 \otimes X_0 \rightarrow X_0 \tag{A.28}$$

In this notation, the Jacobi identity is

$$\ell_2(x_1, \ell_2(x_2, x_3) + \ell_2(x_3, \ell_2(x_1, x_2)) + \ell_2(x_2, \ell_2(x_3, x_1)) = 0 \,. \tag{A.29}$$

We can generalize this construction in small doses. First, consider the underlying vector space to be graded

$$V = \bigoplus_n X_n, \quad n \in \mathbb{N}_0 \tag{A.30}$$

with grading denoted by $\mathrm{x} \equiv \deg(x) = n$ for $x \in X_n$. The grading is introduced in a way which influences the Lie bracket, so that instead of an anticommuting bilinear product, we get a graded-commuting bilinear product

$$\ell_2(x_1, x_2) = (-)^{1+\mathrm{x}_1\mathrm{x}_2} \ell_2(x_2, x_1) \tag{A.31}$$

and the Jacobi identity is generalized accordingly

$$\ell_2(\ell_2(x_1, x_2), x_3) + (-)^{\mathrm{x}_1(\mathrm{x}_2+\mathrm{x}_3)} \ell_2(\ell_2(x_2, x_3), x_1) + (-)^{(\mathrm{x}_2+\mathrm{x}_1)\mathrm{x}_3} \ell_2(\ell_2(x_3, x_1), x_2) = 0 \,. \tag{A.32}$$

In case all vectors would come from X_0 as before, we would return to a (non-graded) Lie algebra.

In the second step, having a series of vector spaces each with a designated degree (grading), we can add a new linear operator

$$\ell_1 : X_n \to X_{n-1}, \quad \ell_1 : X_0 \to 0 \tag{A.33}$$

with the property of being nilpotent $\ell_1^2 = 0$, and respecting the (graded) Leibniz rule

$$\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-)^{\mathrm{x}_1} \ell_2(x_1, \ell_1(x_2)) \,. \tag{A.34}$$

An operator respecting these properties is called a *differential.* A graded Lie algebra equipped with a differential is a *differential graded Lie algebra* and is in itself very useful, but we go further.

The next to final step is to relax the notion of the Jacobi identity holding identically. So far we have defined operators

- ℓ_1 with one input lowering the degree by 1
- ℓ_2 with two inputs that leaves the degree intact

so let's introduce also ℓ_3 with three inputs that raises the degree by 1. The Jacobi identity (A.32) leaves the degree intact and takes three inputs. Let us now allow for (A.32) to hold only up to an element of the same space, which we can achieve by a combination of operators ℓ_1 and ℓ_3.[1] With this generalization, we get

$$\ell_2(\ell_2(x_1, x_2), x_3) + (-)^{\mathrm{x}_1(\mathrm{x}_2+\mathrm{x}_3)} \ell_2(\ell_2(x_2, x_3), x_1) + (-)^{(\mathrm{x}_2+\mathrm{x}_1)\mathrm{x}_3} \ell_2(\ell_2(x_3, x_1), x_2) = \ell_1(\ell_3(x_1, x_2, x_3)) \,. \tag{A.35}$$

Since the generalized Jacobi identity now holds only up to a homotopy (element of the same space), this mathematical structure is called a *differential graded homotopy Lie algebra*, and it is evidently much more general than a Lie algebra.

In the final step, we let loose the possible grading numbers from $\mathbb{N}_0$ to $\mathbb{Z}$, allow for an existence of all operators ℓ_k with $k \in \mathbb{N}$, and establish a tower of identities generalizing (A.34, A.35), which are quadratic in the operators ℓ_k and hold only up to a higher homotopy. Such vastly general structures are named *strongly homotopy Lie algebras* or L_∞ algebras.

[1] To be precise, the identity (A.35) holds only for elements x_k which have $\ell_1(x_k) = 0$, which is not crucial for the motivational presentation. The exact form of this identity is given when the full L_∞ algebra is defined.

A.4 Representations of $ISO(2)$

In the case of massless particles, the little group is $ISO(2)$, isomorphic to the isometry group of a 2D Euclidean plane. Since the faithful representations of this group are less known than the helicity representations, we report on the construction of its unitary irreducible representations following [72]. The generators A, B, J_3 are mapped into Hermitean operators.

Angle basis

The first possibility of building the representation, named the angle basis, starts by picking J_3 with W^2 as the maximal set of commuting operators. Similarly to the known procedure for $\mathfrak{so}(3)$, we define the operators

$$P_\pm = A \pm iB \tag{A.36}$$

which satisfy

$$[J_3, P_\pm] = \pm P_\pm, \quad [P_+, P_-] = 0\,. \tag{A.37}$$

We can rewrite $W^2 = P_+P_- = P_-P_+$ and emphasize that $P_+^\dagger = P_-$. From (4.60) it is readily seen that W^2 has a positive eigenvalue which we denote by μ^2.

Now we choose a simultaneous eigenvector of J_3, W^2 denoted by $|\mu, \sigma\rangle$ and normalized to 1

$$\begin{aligned} W^2 |\mu, \sigma\rangle &= \mu^2 |\mu, \sigma\rangle \\ J_3 |\mu, \sigma\rangle &= \sigma |\mu, \sigma\rangle \ . \end{aligned}$$

Due to the commutation relations we realize that $P_\pm$ act as raising and lowering operators

$$J_3(P_\pm |\mu, \sigma\rangle) = (\sigma \pm 1)(P_\pm |\mu, \sigma\rangle)\,. \tag{A.38}$$

The normalization of raised/lowered states would turn out to be

$$\langle \mu, \sigma \pm 1 | \mu, \sigma \pm 1\rangle = \langle \mu, \sigma| P_\pm^\dagger P_\pm |\mu, \sigma\rangle = \langle \mu, \sigma| W^2 |\mu, \sigma\rangle = \mu^2 \tag{A.39}$$

A special possibility is $\mu^2 = 0$. In that case, we recover the usual one-dimensional unitary representation for massless particles since then $P_\pm |0, \sigma\rangle = 0$. The only remaining generator acting non-trivially is J_3 whose eigenvalues correspond to helicity $J_3 |0, \sigma\rangle = \sigma |0, \sigma\rangle$.

For $\mu^2 \neq 0$, we define a new normalization so that all states are normalized to unity

$$|\mu, \sigma \pm 1\rangle \equiv P_\pm |\mu, \sigma\rangle \left(\pm \frac{i}{\mu}\right) . \tag{A.40}$$

A very important feature in this case is that the representation is necessarily infinite-dimensional; starting with some σ_0, say $\sigma_0 = 0$, we can construct infinitely many new vectors with helicity $\sigma = 0, \pm 1, \pm 2, ...$ within a single value of μ^2. This is the reason that particles corresponding to this representations are named "infinite-spin" particles. The matrix elements of the generators in this representation are

$$\begin{aligned}\langle \mu, \sigma'| J_3 |\mu, \sigma\rangle =& \sigma \delta^{\sigma}{}_{\sigma'}\\ \langle \mu, \sigma'| P_\pm |\mu, \sigma\rangle =& \mp i\mu \delta^{\sigma'}{}_{\sigma \pm 1}\end{aligned}$$

By exponentiation it can be seen that the representation matrices for a finite transformation are given by [72]:

$$D^\mu(b_A, b_B, \theta)^{m'}{}_m = D^\mu(\mathbf{b}, \theta)^{m'}{}_m = e^{i(m-m')\phi} \mathbf{J}_{m-m'}(\mu b) e^{-im\theta} \tag{A.41}$$

The three parameters (b_A, b_B, θ), each correspond to one of the operators A, B, J_3. The first two parameters can be written as a 2-vector in Euclidean space $\mathbf{b} = (b, \phi)$ in "polar" coordinates. $\mathbf{J}_n$ is the Bessel function of the first kind.

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\alpha+1)} \left(\frac{x}{2}\right)^{2j+\alpha} \tag{A.42}$$

Another representation valid for integer n is

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin\tau - n\tau)}\, d\tau \tag{A.43}$$

Plane Wave basis

The other possible basis comes from choosing A, B with W^2 as a maximal set of commuting operators and building the representation with their eigenvectors. This can be done with the method of induced representations. The two generators A, B can be written as a vector $\mathbf{T} = (A, B)$. We can then choose a standard vector of their eigenvalues as $\vec{\mu}_0 = (\mu, 0)$. There is only one independent eigenstate of $\mathbf{T}$ since it does not commute with J_3

$$A |\vec{\mu}_0\rangle = \mu |\vec{\mu}_0\rangle \tag{A.44}$$

$$B |\vec{\mu}_0\rangle = 0 \tag{A.45}$$

$$W^2 |\vec{\mu}_0\rangle = \mu^2 |\vec{\mu}_0\rangle \tag{A.46}$$

Due to the commutation relations of A, B with J_3 it is visible that $\mathbf{T}$ transforms as a Euclidean vector under rotations $R(\theta) = e^{-iJ\theta}$, so

$$e^{-i\theta J} T_k e^{i\theta J} = T_m R(\theta)^m{}_k \tag{A.47}$$

Thus $T_k R(\theta) |\vec{\mu}_0\rangle = \mu_k R(\theta) |\vec{\mu}_0\rangle$, where μ_k is the k-th component of the vector obtained by rotating $\vec{\mu}_0$. This way we can build the entire vector space spanning the irreducible sector with μ^2

$$|\vec{\mu}\rangle = R(\theta) |\vec{\mu}_0\rangle \tag{A.48}$$

The set of vectors (continuous set, since $\vec{\mu}$ is a vector with two continuous parameters - μ and θ) $\{|\vec{\mu}\rangle\}$ forms the irreducible vector space. Here, and in the fact that μ^2 can take on arbitrary real values we find the origin of the name "continuous spin" particle. The representation of the group can again be obtained by exponentiation:

$$T(\mathbf{b}) |\vec{\mu}\rangle = e^{-i\mathbf{b}\cdot\vec{\mu}} |\vec{\mu}\rangle \tag{A.49}$$

$$R(\phi) |\vec{\mu}\rangle = |\vec{\mu}'\rangle \tag{A.50}$$

where $\vec{\mu}' = R(\phi)\vec{\mu} = (\mu, \theta + \phi)$. The ortonormality condition can be chosen as

$$\langle \vec{\mu}' | \vec{\mu} \rangle = \langle \mu, \theta' | \mu, \theta \rangle = 2\pi\delta(\theta' - \theta) \tag{A.51}$$

The vectors are orthogonal if $\vec{\mu}$ are different in length and when they are equal length but of different angle.

Appendix B

Details for calculations in MHS theory

B.1 Second order (metric like) gauging

In chapter 2 we have performed the gauging procedure and introduced the gauge potential by

$$u_a \to u_a + h_a(x,u) \tag{B.1}$$

This is not the only possibility, and here we display a different approach which, as it turns out, leads to a composite field. In case where $m = 0$ for the scalar field action (2.20) written as

$$S[\phi] = \int d^d x \, \frac{d^d u}{(2\pi)^d} \, u^2 \star W_\phi(x,u) \tag{B.2}$$

we have a symmetry for a rigid gauge parameter $\varepsilon(u)$ [1]

$$\delta S = i \int d^d x \, \frac{d^d u}{(2\pi)^d} \left([\varepsilon(u) \overset{\star}{,} W_\phi(x,u) \star u^2] + W_\phi(x,u) \star [u^2 \overset{\star}{,} \varepsilon(u)] \right) = 0 \; . \tag{B.3}$$

As the Moyal commutator is a total derivative both in x and u variables, the first term is only a boundary term. The second term has a Moyal commutator of only u-dependent quantities, so it vanishes.

Considering a local symmetry $\varepsilon = \varepsilon(x,u)$, the variation of the free field action is non-vanishing

$$\delta S = i \int d^d x \, \frac{d^d u}{(2\pi)^d} \, W_\phi(x,u) \star [u^2 \overset{\star}{,} \varepsilon(x,u)] \; . \tag{B.4}$$

To ensure symmetry is preserved, in the spirit of Yang Mills theory, this calls for a compensating field

$$u^2 \to u^2 - h(x,u) \; . \tag{B.5}$$

[1] To see this, add and subtract $W_\phi(x,u) \star u^2 \star \varepsilon(u)$ under the integral.

To first order in $\varepsilon(x,u)$, by neglecting total derivatives under the integral, we have

$$\delta S = \int d^d x\, d^d u\, W_\phi \star (-\delta h + i[(u^2 - h) \overset{\star}{,} \varepsilon]) \ . \tag{B.6}$$

To keep the action symmetric under local transformations, we can infer that $h(x,u)$ must transform as

$$\delta h(x,u) = 2u \cdot \partial^x \varepsilon(x,u) - i[h(x,u) \overset{\star}{,} \varepsilon(x,u)] \tag{B.7}$$

reproducing the result in [43]. As we have shown above in chapter 6, the field $h(x,u)$ is actually not fundamental, instead given by

$$h(x,u) = 2u^a h_a(x,u) + h_a(x,u) \star h^a(x,u) \tag{B.8}$$

B.2 Details for the interaction with simple matter

As we have seen in chapter 6, ordinary matter couples to the MHS field through a minimal coupling of simple currents to a tower of HS fields. Here we provide the details of that calculation. The interaction term (6.5) is

$$S_m^{(\mathrm{int})}[\phi, h] = \int d^d x\, d^d u\, \big(\phi_r^*(x) \star K_{\mathrm{int}}^{rs}(x,u) \star \phi_s(x)\big)\delta^d(u) \ . \tag{B.9}$$

In the geometric phase we have $e_a(x,u) = u_a + h_a(x,u)$. Following (6.3), the interacting part is given by

$$K_{\mathrm{int}}(x,u) = h(x,u) = 2\, u^a h_a(x,u) + h_a(x,u) \star h^a(x,u) \tag{B.10}$$

where $h(x,u)$ is a composite object obtained from the MHS potential, already introduced in (2.97). If we now use a Taylor expansion in the auxiliary coordinates, and a compactified notation

$$h(x,u) = \sum_{s=0}^{\infty} h_{(s)}^{\mu_1 \cdots \mu_s}(x)\, u_{\mu_1} \cdots u_{\mu_s} = \sum_{s=0}^{\infty} h_{(s)}^{\mu_1 \cdots \mu_s}(x)\, (u_\mu)^s \tag{B.11}$$

we can insert this into (B.9) and easily integrate over the auxiliary space, facilitated by the presence of the Dirac delta function. We will use the integral representation of the Moyal product (A.13). We start by

$$h(x,u) \star \varphi(x) = \int d^d z\, \frac{d^d v}{(2\pi)^d} e^{-izv} h(x, u+v) \varphi\left(x + \frac{z}{2}\right) \tag{B.12}$$

and

$$\varphi^*(x)\star h(x,u)\star\varphi(x)=\int d^dy'd^dz\,\frac{d^dw'}{(2\pi)^d}\frac{d^dv}{(2\pi)^d}e^{iy'w'-izv}\varphi^*\left(x+\frac{y'}{2}\right)h(x,u+v+w')\varphi\left(x+\frac{z}{2}\right)\,.$$

We now integrate over the auxiliary space u with the delta function present

$$\int d^du\,\varphi^*(x)\star h(x,u)\star\varphi(x)\delta^{(d)}(u)=\int d^dy'd^dz\,\frac{d^dw'}{(2\pi)^d}\frac{d^dv}{(2\pi)^d}e^{iy'w'-izv}\varphi^*\left(x+\frac{y'}{2}\right)h(x,v+w')\varphi\left(x+\frac{z}{2}\right)$$

If we now employ the expansion (B.11), the integral above becomes

$$\begin{aligned}
&\int d^dy'd^dz\,\frac{d^dw'}{(2\pi)^d}\frac{d^dv}{(2\pi)^d}\sum_{s=0}^{\infty}h_{(s)}^{\mu_1\cdots\mu_s}(x)\,(v_\mu+w'_\mu)^se^{iy'w'-izv}\varphi^*\left(x+\frac{y'}{2}\right)\varphi\left(x+\frac{z}{2}\right)\\
=&\int d^dy'd^dz\,\frac{d^dw'}{(2\pi)^d}\frac{d^dv}{(2\pi)^d}\sum_{s=0}^{\infty}h_{(s)}^{\mu_1\cdots\mu_s}(x)\sum_{k=0}^{s}\binom{s}{k}(v_\mu)^k(w'_\mu)^{s-k}e^{iy'w'-izv}\varphi^*\left(x+\frac{y'}{2}\right)\varphi\left(x+\frac{z}{2}\right)\\
=&\int d^dy'd^dz\,\frac{d^dw'}{(2\pi)^d}\frac{d^dv}{(2\pi)^d}\sum_{s=0}^{\infty}h_{(s)}^{\mu_1\cdots\mu_s}(x)\sum_{k=0}^{s}\binom{s}{k}(i)^s(-1)^k\left(\partial^z_\mu\right)^k\varphi\left(x+\frac{z}{2}\right)\left(\partial^{y'}_\mu\right)^{s-k}\varphi^*\left(x+\frac{y'}{2}\right)e^{iy'w'-izv}\\
=&\sum_{s=0}^{\infty}h_{(s)}^{\mu_1\cdots\mu_s}(x)\sum_{k=0}^{s}\binom{s}{k}\left(\frac{i}{2}\right)^s(-1)^k\left(\partial^x_\mu\right)^k\varphi(x)\left(\partial^x_\mu\right)^{s-k}\varphi^*(x)
\end{aligned}\tag{B.13}$$

The interacting part of the action is thus given by

$$S_m^{(\mathrm{int})}[\varphi,h]=\sum_{s=0}^{\infty}\int d^dx\,J^{(s)}_{\mu_1\cdots\mu_s}(x)\,h_{(s)}^{\mu_1\cdots\mu_s}(x)\,,\tag{B.14}$$

where the spin-s currents are of the form

$$J^{(s)}_{\mu_1\cdots\mu_s}(x)=\frac{i^s}{2^s}\sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\partial^x_\mu\right)^k\varphi(x)\left(\partial^x_\mu\right)^{s-k}\varphi^*(x)\tag{B.15}$$

$$=\frac{i^s}{2^s}\,\varphi(x)^*\overset{\leftrightarrow}{\partial}_{\mu_1}\cdots\overset{\leftrightarrow}{\partial}_{\mu_s}\varphi(x)\,.\tag{B.16}$$

B.3 Curvature tensor measures triviality

Here we prove that the HS field strength measures the triviality of HS configurations, i.e.

$$\text{HS master space field is pure gauge}\qquad\Longleftrightarrow\qquad F_{ab}(x,u)=0\tag{B.17}$$

in a domain of configurations containing $h_a(x,u)=0$.

Proof. From (2.50) it follows directly that the theorem is valid in the linear approximation, since the linear term in (2.84) can be interpreted as the exterior derivative of a form $h_a^{\mu_1\ldots\mu_n}(x)$ if Greek indices μ_j are treated as internal. It means that in the linearized theory if and only if $F_{ab}(x,u)=0$ in a ball, there exists a master space function $\varepsilon(x,u)$ such that

$$h_a(x,u)=\partial_a\,\varepsilon(x,u) \tag{B.18}$$

in the same ball. But this is just the linearized pure gauge condition for the MHS potential $h_a(x,u)$.

Let us extend this to large fields. First we prove the left-to-right arrow in (B.17). If the MHS vielbein field is pure gauge, then by (2.66) we can write it as

$$e_a(x,u)=e_\star^{-i\,\mathcal{E}(x,u)}\star u_a\star e_\star^{i\,\mathcal{E}(x,u)}=u_a-ie_\star^{-i\,\mathcal{E}(x,u)}\star\partial_a^x e_\star^{i\,\mathcal{E}(x,u)} \tag{B.19}$$

so a pure gauge MHS master field $h_a(x,u)$ is of the form

$$h_a(x,u)=-ie_\star^{-i\,\mathcal{E}(x,u)}\star\partial_a^x e_\star^{i\,\mathcal{E}(x,u)}\ . \tag{B.20}$$

Plugging this into (2.50), and using the identities

$$e_\star^{-i\,\mathcal{E}(x,u)}\star e_\star^{i\,\mathcal{E}(x,u)}=1\qquad,\qquad e_\star^{-i\,\mathcal{E}(x,u)}\star\partial_a^x e_\star^{i\,\mathcal{E}(x,u)}=-\partial_a^x e_\star^{-i\,\mathcal{E}(x,u)}\star e_\star^{i\,\mathcal{E}(x,u)} \tag{B.21}$$

we conclude that MHS field strength $F_{ab}(x,u)$ vanishes for pure gauge HS fields. Therefore,

$$\text{HS phase space field is pure gauge}\qquad\Longrightarrow\qquad F_{ab}(x,u)=0\ . \tag{B.22}$$

Proving the opposite direction of (B.17) happens to be more involved. We want to find the general solution of the equation

$$F_{ab}(x,u)=0\ . \tag{B.23}$$

To do this, let us start from the linearized solution (B.18) and build a full solution by a formal perturbative series[2]

$$h_a(x,u)=\sum_{n=1}^{\infty}\Delta_a^{(n)}(x,u)\ . \tag{B.24}$$

Introducing (B.24) into (B.23), using (2.50), and collecting the terms of the same order, we obtain

$$\partial_a^x\Delta_b^{(n)}(x,u)-\partial_b^x\Delta_a^{(n)}(x,u)=-i\sum_{r=1}^{n-1}[\Delta_a^{(r)}(x,u)\stackrel{\star}{,}\Delta_b^{(n-r)}(x,u)]\ . \tag{B.25}$$

[2]In this construction it is not assumed that the HS potential $h_a(x,u)$ is small. We can introduce a formal parameter θ, and consider (B.24) as an expansion in θ. Eventually we put $\theta\to 1$.

We see that it has a form which can be attacked by mathematical induction. For $n = 1$ it becomes

$$\partial_a^x \Delta_b^{(1)}(x,u) - \partial_b^x \Delta_a^{(1)}(x,u) = 0 \tag{B.26}$$

for which the general solution is

$$\Delta_a^{(1)}(x,u) = \partial_a^x \mathcal{E}(x,u) \tag{B.27}$$

where $\mathcal{E}(x,p)$ is an arbitrary function. For $n = 2$ we get

$$\begin{aligned}
\partial_a^x \Delta_b^{(2)} - \partial_b^x \Delta_a^{(2)} &= -\, i\, [\Delta_a^{(1)} \stackrel{\star}{,} \Delta_b^{(1)}] \\
&= -\, i\, [\partial_a^x \mathcal{E} \stackrel{\star}{,} \partial_b^x \mathcal{E}] \\
&= -\, \frac{i}{2}\, (\partial_a^x [\mathcal{E} \stackrel{\star}{,} \partial_b^x \mathcal{E}] - \partial_b^x [\mathcal{E} \stackrel{\star}{,} \partial_a^x \mathcal{E}])
\end{aligned} \tag{B.28}$$

for which the solution is

$$\Delta_b^{(2)}(x,u) = -\frac{i}{2}\, [\mathcal{E}(x,u) \stackrel{\star}{,} \partial_a^x \mathcal{E}(x,u)] + \partial_a^x \mathcal{E}'(x,u)\ . \tag{B.29}$$

The trivial exact part of the solution, which appears at every order, is of the same form as the first-order solution; it introduces nothing new and can therefore be ignored in the construction of the general solution. Now we conjecture that the generic solution is given by

$$\begin{aligned}
\Delta_a^{(n)} =& \frac{(-i)^{n-1}}{n!}\, [\mathcal{E} \stackrel{\star}{,} [\mathcal{E} \stackrel{\star}{,} \ldots [\mathcal{E} \stackrel{\star}{,} \partial_a \mathcal{E}]] \ldots] \qquad (n-1\ \text{ Moyal brackets}) \\
=& \frac{(-i)^{n}}{n!}\, [\mathcal{E} \stackrel{\star}{,} [\mathcal{E} \stackrel{\star}{,} \ldots [\mathcal{E} \stackrel{\star}{,} u_a]] \ldots] \qquad (n\ \text{ Moyal brackets})\ .
\end{aligned} \tag{B.30}$$

This can be proven by induction. Using (B.30) in (B.24) gives us finally

$$\begin{aligned}
h_a(x,u) =& \sum_{n=1}^{\infty} \frac{(-i)^n}{n!}\, [\mathcal{E}(x,u) \stackrel{\star}{,} [\mathcal{E}(x,u) \stackrel{\star}{,} \ldots [\mathcal{E}(x,u) \stackrel{\star}{,} u_a]] \ldots] \\
=& e_\star^{-i\,\mathcal{E}(x,u)} \star u_a \star e_\star^{i\,\mathcal{E}(x,u)} - u_a \\
=& -\, i\, e_\star^{-i\,\mathcal{E}(x,u)} \star \partial_a^x e_\star^{i\,\mathcal{E}(x,u)}
\end{aligned} \tag{B.31}$$

where to pass from the first to the second line we used the Baker-Campbell-Hausdorff lemma. We have obtained (B.20), therefore we proved that

$$F_{ab}(x,u) = 0 \qquad \Longrightarrow \qquad \text{HS potential is a pure gauge} \tag{B.32}$$

in some neighbourhood of $h_a(x,u) = 0$.

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Curriculum vitae

Mateo Paulišić was born on the 24th of September 1991 and grew up in Pazin. After a classical grammar school in "Pazinski kolegij" he enrolled into an integrated reasearch study of physics at the Faculty of Science, University of Zagreb. His studies were successfully finished in 2016 with a master thesis "Spacetime singularities" created under the supervision of assist. prof. Ivica Smolić.

In October 2016, he was employed as an assistant by the Department of Physics, University of Rijeka, working with the advisor prof. Predrag Dominis Prester, and in 2018 enrolled in the doctoral study at the Department of Physics, University of Rijeka. He participated at conferences and schools in Zagreb, London, Trieste, Bruxelles, Sofia and Vienna.

In his private time, he is proud to have been a member of Klapa Teran, Pižolot and Pinguentum as well as bands Oridano Gypsy Jazz Band and Hot Club de Istra.

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